Axial Tilt & Torsion: An Orientation Representation for Four-Cable Suspended Parallel Robots

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Abstract This work introduces an orientation representation for platforms suspended by four cables, expressed through a tilt angle around a horizontal axis and an independent torsion about the platform normal. The proposed representation provides an explicit description of both the direction and the amplitude of inclination, in contrast with classical Euler angles, which offer limited geometric interpretation. Orientation is expressed as the product of two rotation matrices defining the tilt axis and the torsion. The method is applied to the static equilibrium workspace of a suspended parallel robot. The simulation results show that the representation yields a more intuitive and symmetry-consistent description of platform inclination while preserving full rotational equivalence with classical representations.

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1 Introduction

Classical rotation parameterizations such as Euler angles, rotation matrices, unit quaternions, and rotation vectors, each present intrinsic limitations: Euler angles are compact but suffer from singularities, quaternions avoid singularities but require normalization, and rotation vectors remain nonlinear despite their intuitive local interpretation [1]. As a result, the choice of representation is often dictated by the specific mechanical or geometric constraints of the system under study.

For parallel mechanisms, alternative representations have been proposed to improve the geometric interpretability of orientation. Among them, the Tilt and Torsion (T&T) formulation introduced by Bonev et al. [2] decomposes orientation into a tilt that aligns the vertical axis with its final direction and a torsion about the body normal. This representation has proved effective for analyzing constrained architectures and orientation workspaces. However, in the classical T&T formulation, the tilt is defined through the orientation of a sagittal plane, and the horizontal axis about which the tilt occurs is not explicitly parameterized. As a consequence, the direction of inclination is only indirectly encoded.

Cable-suspended parallel robots (CSPRs) constitute a class of mechanisms for which the direction of inclination is as critical as its magnitude. The platform is maintained in equilibrium by gravity and cable tensions, and excessive tilt in a specific direction may lead to loss of equilibrium or overturning [3]. For these systems, it is therefore essential to describe not only how much the platform tilts, but also around which horizontal axis this tilt occurs. Classical Euler angles and existing T&T representations do not provide this information explicitly.

This paper introduces an orientation representation tailored to suspended platforms, based on a decomposition into a tilt about an explicitly defined horizontal axis and an independent torsion about the platform normal. By directly parameterizing (i) the orientation of the tilt axis, (ii) the tilt amplitude, and (iii) the torsion, the proposed Axial Tilt and Torsion (AT&T) formulation yields a physically meaningful and symmetry-consistent description of inclination. Unlike classical T&T, AT&T makes the tilt axis itself an explicit parameter, which is particularly relevant for analyzing stability and cable tension distribution in CSPRs.

The proposed representation is applied to a four-cable suspended parallel robot, and its properties are illustrated through the analysis of the static equilibrium workspace. The results show that AT&T preserves full rotational equivalence with Euler angles while offering clearer geometric interpretation, improved symmetry behavior, and more intuitive visualization of inclination direction, features that are essential for the study and control of cable-suspended systems.

2 Notations

The following notations are used in this work:

- $\mathbf{u_i}$, unit vector u with index i.
- $\mathcal{F}_i$, frame with index i, defined by the right-handed triad $(\mathbf{x_i}, \mathbf{y_i}, \mathbf{z_i})$.
- $\mathbf{v} = (a, b, c)_{\mathcal{F}_i}$, the components a, b, and c of $\mathbf{v}$ in $\mathcal{F}_i$, respectively along $\mathbf{x_i}$, $\mathbf{y_i}$, and $\mathbf{z_i}$.
- $\mathbf{R}_{mat}$, the rotation matrix named *mat*. $\mathbf{R}_u(a)$ denotes a rotation matrix of amplitude a about axis $\mathbf{u}$.
- S, C, and T stand for sine, cosine and tangent.
- AS, AC, and AT stand for arcsine, arccosine and arctangent.

3 Rotation in terms of an inclination and a swivel

For a given rotation $\mathbf{R}$, there exists an infinite family of rotations $\mathbf{R}_u(\theta)$ that map $\mathbf{z_b}$ onto $\mathbf{z_i} = \mathbf{z_p}$, each associated with its own rotation $\mathbf{R}_z(\sigma)$ (Fig. 1). Some of these $\mathbf{R}_u(\theta)$ are equivalent to the first two rotations in Euler angle parameterizations that end with a rotation about the Z-axis.

The shortest rotation between two unit vectors is a rotation about the normal to these two vectors by the circumscribed angle. It means θ reaches its minimum when $\mathbf{u}$ is aligned with $\mathbf{z_b} \times \mathbf{z_p}$. In this study, since $\mathbf{z_b}$ is vertical, $\mathbf{u}$ is horizontal.

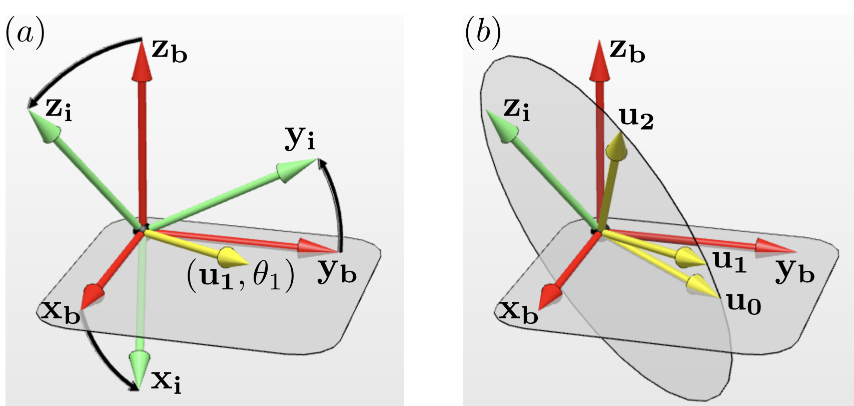

Fig. 1 (a) Rotation of $\mathcal{F}_b$ by an angle θ_1 around $\mathbf{u_1}$ to obtain $\mathcal{F}_i$. (b) All vectors $\mathbf{u}$ such that $\mathbf{R}_u(\theta)$ maps $\mathbf{z_b}$ onto $\mathbf{z_i}$ lie on a disc $\mathcal{D}$.

The inclination therefore reduces to tilt, corresponding to the special case described by Bonev et al. as "Tilt and Torsion" (T&T angles). Expressing the inclination through a tilt is especially advantageous, as it provides a significantly simpler and more intuitive analysis of orientation compared with Euler angles.

4 Axial Tilt and Torsion (AT&T)

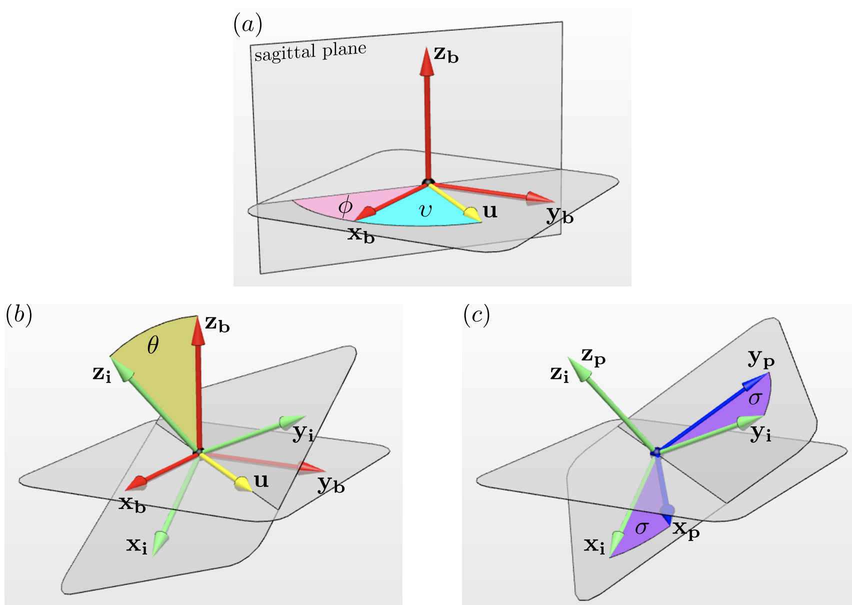

Fig. 2 Orientation of $\mathcal{F}_p$ with respect to $\mathcal{F}_b$ expressed using T&T angles versus AT&T angles. Step (a): positioning of the sagittal plane (SP) using ϕ (T&T) versus positioning of $\mathbf{u}$ using υ (AT&T). Step (b): rotation of $\mathcal{F}_b$ by an angle θ with $\mathbf{z_i} \in (SP)$ and $\langle \mathbf{z_b}, \mathbf{z_i} \rangle = \theta$ (T&T) versus rotation of $\mathcal{F}_b$ by an angle θ about $\mathbf{u}$ (AT&T), defining the tilt. Step (c): rotation of $\mathcal{F}_i$ by an angle σ about $\mathbf{z_i}$, defining the torsion. In AT&T, the complete orientation is obtained as $\mathcal{F}_b \xrightarrow{\mathbf{R}_u} \mathcal{F}_i \xrightarrow{\mathbf{R}_z} \mathcal{F}_p$, yielding $\mathbf{R}_{\text{AT\&T}} = \mathbf{R}_u(\theta)\mathbf{R}_z(\sigma)$.

Defining an orientation through two successive rotations—the first about an axis lying in the horizontal plane and the second about the body's normal—is known as the T&T (Tilt and Torsion) representation. In the T&T framework, the parameterization does not directly describe the tilt axis; instead, it specifies the solid's *sagittal plane*, within which the tilt motion occurs.

To avoid any ambiguity between the classical T&T representation and the formulation proposed here, we introduce the AT&T (Axial Tilt and Torsion) representation (Fig. 2), with the following parameters: υ (from upsilon), describing the orientation of $\mathbf{u}$; θ, the t̲ilt angle; and σ, the torsion (also referred to as s̲wivel).

Mathematically, the AT&T rotation matrix can be expressed as

$$\mathbf{R}_{\text{AT\&T}} = \underbrace{\begin{bmatrix} C_\upsilon^2 + S_\upsilon^2 C_\theta & C_\upsilon S_\upsilon (1 - C_\theta) & S_\upsilon S_\theta \\ C_\upsilon S_\upsilon (1 - C_\theta) & S_\upsilon^2 + C_\upsilon^2 C_\theta & -C_\upsilon S_\theta \\ -S_\upsilon S_\theta & C_\upsilon S_\theta & C_\theta \end{bmatrix}}_{\text{Axial Tilt}} \underbrace{\begin{bmatrix} C_\sigma & -S_\sigma & 0 \\ S_\sigma & C_\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Torsion}}. \tag{1}$$

Since $\mathbf{R}_u(\theta) = \mathbf{R}_{-u}(-\theta)$, therefore

$$\mathbf{R}_{\text{AT\&T}} = \mathbf{R}_u(\theta)\mathbf{R}_z(\sigma) = \mathbf{R}_{-u}(-\theta)\mathbf{R}_z(\sigma). \tag{2}$$

The choice between these two equivalent parameterizations depends on the definition domains selected for υ and θ.

The following remarks provide further insight into its behavior in special configurations.

Remark 1

With the AT&T parametrization, a mathematical singularity arises when the solid is horizontal (without inclination). In this configuration, one may state that it is inclined by 0° with respect to any vector $\mathbf{u}$ lying in the plane $(\mathbf{x_b}, \mathbf{y_b})$, and therefore for any value of υ.

Remark 2

If one assumes $\mathbf{R}_u(\theta) = \mathbf{R}_{z_b}(\upsilon)\mathbf{R}_x(\theta)\mathbf{R}_{z_i}(-\upsilon)$ then

$$R_{\text{AT\&T}} = \underbrace{\begin{bmatrix} C_\upsilon & -S_\upsilon & 0 \\ S_\upsilon & C_\upsilon & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & -S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix}}_{\text{places } \mathbf{z_i} \text{ on } \mathcal{F}_b \text{ with } \mathbf{z_p} = \mathbf{z_i}} \underbrace{\begin{bmatrix} C_{(\sigma-\upsilon)} & -S_{(\sigma-\upsilon)} & 0 \\ S_{(\sigma-\upsilon)} & C_{(\sigma-\upsilon)} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{turns around } \mathbf{z_p}} \tag{3}$$

The AT&T angles may be viewed as a modified form of ZXZ Euler angles, where the parameters exhibit coupling.

4.1 Transition laws between Euler XYZ and AT&T angles

The comparison between AT&T reference values and XYZ Euler measurements relies on conversion laws directly resulting from the equivalence between $\mathbf{R}_{AT\&T}$ and $\mathbf{R}_{XYZ\,Euler}$. Table 1 gives the formulation for converting AT&T to Euler XYZ and vice versa.

Table 1 Conversion formulas between AT&T angles and XYZ Euler angles.

<table>
<tr><th>AT&T → XYZ Euler</th><th>XYZ Euler → AT&T</th></tr>
<tr><td>$\alpha = AT(C_\upsilon T_\theta)$</td><td>$\upsilon = AT\left(\frac{T_\beta}{S_\alpha}\right) \mod (\pi)$
with $\text{sign}(S_\upsilon) = \text{sign}(S_\beta)$</td></tr>
<tr><td>$\beta = AS(S_\upsilon S_\theta)$</td><td>$\theta = AC\left(C_\alpha C_\beta\right)$</td></tr>
<tr><td>$\gamma = AT\left(\frac{C_\upsilon S_{(\sigma-\upsilon)} + S_\upsilon C_\theta C_{(\sigma-\upsilon)}}{C_\upsilon C_{(\sigma-\upsilon)} - S_\upsilon C_\theta S_{(\sigma-\upsilon)}}\right)$</td><td>$\sigma = AS\left(\frac{S_\alpha S_\beta C_\gamma + (C_\alpha + C_\beta)S_\gamma}{1 + C_\alpha C_\beta}\right)$</td></tr>
<tr><td colspan="2">Domains of definitions:
$\alpha \in [-90°, 90°]$, $\beta \in [-90°, 90°]$, and $\gamma \in [-45°, 45°]$
$\upsilon \in [0°, 360°]$, $\theta \in [0°, 90°]$, and $\sigma \in [-45°, 45°]$</td></tr>
</table>

5 Results and Discussion

The behavior of the proposed AT&T angles was evaluated by comparing them with classical XYZ Euler angles across the static equilibrium workspace (SEW) of the four-cable CSPR. The static equilibrium conditions are obtained from the six equations of the fundamental principles of statics, supplemented by an objective function: minimization of the potential energy [4, 5]. The platform's center of gravity was positioned at a height of 1500 mm, and equilibrium configurations were computed for all feasible horizontal positions within the structure.

5.1 Representation of AT&T angles

Figure 3 shows the evolution of AT&T angles for the equilibrium configurations considered in the study. The parameter υ (Fig. 3a) identifies the orientation of the horizontal tilt axis, while θ (Fig. 3b) gives the tilt amplitude. When visualized together (Fig. 3c), these two parameters form a vector field that directly conveys both the magnitude and direction of inclination. Hence, the AT&T representation makes the tilt direction explicit. This representation provides a more intuitive understanding of how the platform responds to variations in cable geometry and gravitational loading.

In contrast, the inclination amplitude (calculated using Euler angles), $inc = |AC(C_\alpha C_\beta)|$ provides a scalar measure of the angle between $\mathbf{z_b}$ and

$\mathbf{z_p}$ but does not reveal the direction in which the platform is inclined. As a result, the supporting axis of the inclination cannot be inferred directly from Euler angles, even though this information is essential for interpreting the platform's stability and for designing corrective actions.

To confirm that the choice of orientation parameterization does not influence the static analysis, the equilibrium configurations were computed using both XYZ Euler angles and the proposed AT&T angles. The resulting rotation matrices and cable tensions were identical up to numerical precision, indicating that AT&T preserves the underlying equilibrium conditions while simply reorganizing the orientation information into a more interpretable form.

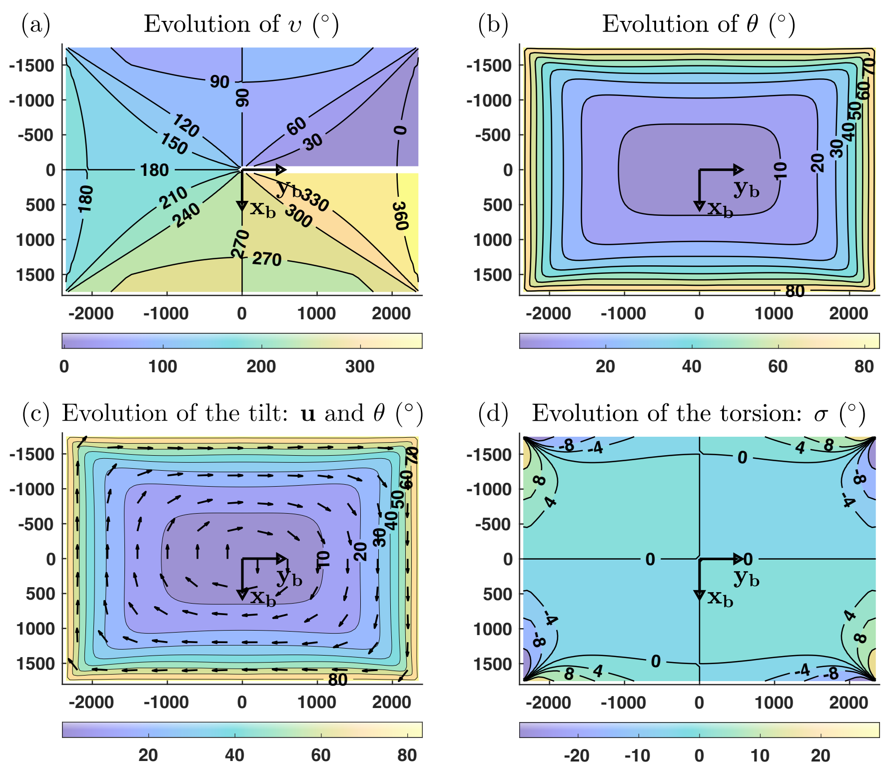

Fig. 3 Evolution of the AT&T angles. In (c), the tilt is visualized by superposing the supporting direction, represented by the unit vector $\mathbf{v}$ (inclined by υ from $\mathbf{x_b}$ in $\mathcal{F}_b$), with the tilt amplitude θ.

Although the AT&T representation becomes underdetermined when the platform is perfectly horizontal ($\theta = 0$), this does not affect the static analysis. In this configuration, any horizontal axis can serve as the tilt axis, the rotation matrix and the torsion angle remain well defined, and the equilibrium computation is unaffected. In practice, the platform reaches $\theta = 0$ only at isolated points of the workspace, and the tilt direction is physically irrelevant in such configurations.

To further illustrate the behavior of both Euler and AT&T angles, Fig. 4 presents the evolution of orientation along a straight-line trajectory between two workspace points, denoted M and N. As seen in Fig. 4a, at point I (located at 50% of the trajectory), the platform is horizontal resulting in $\alpha = \beta = 0$. In this case, due to the symmetry of the platform and the working environment, we also have $\gamma = 0$ (Fig. 4b). This is a point of singularity in case of XYZ Euler angles. The evolution of AT&T angles are shown in Fig. 4c. At I, it can be seen that there is discontinuity in the values of υ and θ due to the positive angle constraints imposed for their calculation. However, the tilt is continuous and does not affect the static equilibrium of the platform.

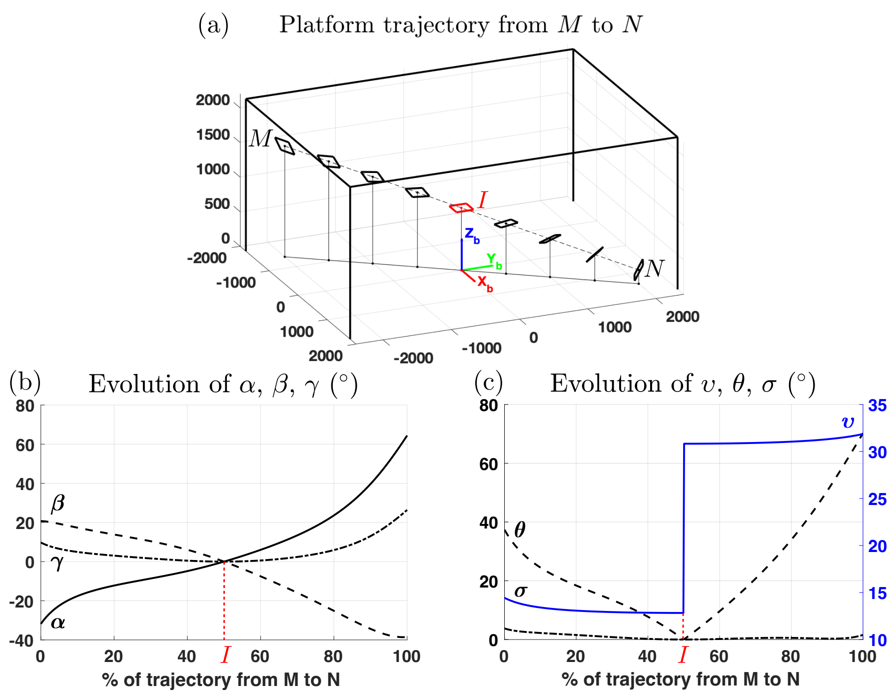

Fig. 4 Evolution of XYZ Euler angles and AT&T angles during a linear trajectory from $M = (-1400, -2000, 1600)_{\mathcal{F}_b}$ to $N = (1400, 200, 200)_{\mathcal{F}_b}$ such that $[MN]$ intersects $\mathbf{z_b}$. At this intersection point (named I) the platform is horizontal as shown in red on subfigure *a*. It's the special case of the singularity explained in *Remark 1* section 4.

5.2 Behavior of torsion in symmetric configurations

A notable difference emerges when examining torsion (Fig. 5). For a square platform inside a square-section structure, the expected physical behavior is symmetric with respect to the diagonals. However, the Euler torsion γ does not exhibit this symmetry. This asymmetry arises from the coupling between Euler angles and the non-orthogonality of their parameterization with respect to the physical geometry.

In contrast, the AT&T torsion σ displays perfect symmetry with respect to the structural diagonals. This behavior aligns with the inherent symmetry of the CSPR and reflects the fact that AT&T isolates torsion from tilt. As a result, σ varies smoothly and predictably across the workspace, making it a more natural descriptor for analyzing orientation in symmetric suspended systems.

Overall, the results demonstrate that AT&T angles preserve the full rotational information of Euler XYZ angles while offering a representation that is more consistent with the physical behavior of CSPRs and more suitable for interpretation, visualization, and control.

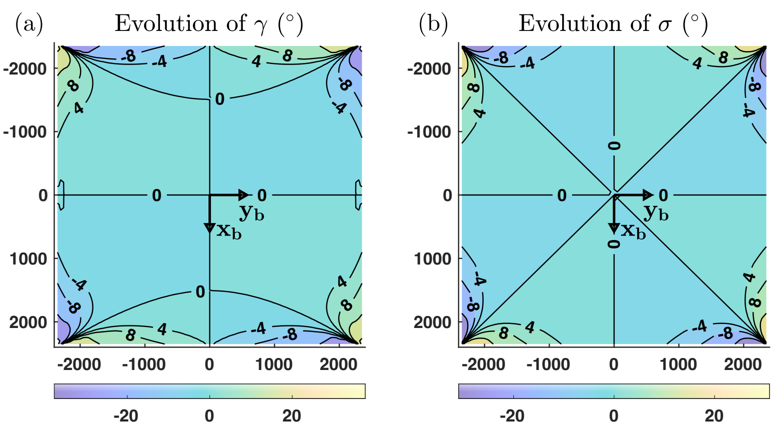

Fig. 5 Evolution of the torsion represented by γ (XYZ Euler) and by σ (AT&T).

6 Conclusion

This work introduced the Axial Tilt and Torsion (AT&T) representation as a structured framework for describing the orientation of a four-cable suspended parallel robot. By explicitly defining the tilt axis, tilt amplitude, and torsion, AT&T provides a clearer and more physically meaningful characterization of inclination than classical Euler angles, while preserving full rotational equivalence and comparable computational cost. The analysis of the static equilibrium workspace demonstrated that AT&T yields more interpretable and consistent orientation parameters, particularly in configurations governed by gravitational constraints. These results indicate that AT&T is a suitable and effective representation for studying and controlling the orientation of CSPRs. Future work will extend this formulation to dynamic modeling and closed-loop control.

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