Closed Form Solution of the Inverse Kinematics Equations on the Hyper-State of 3C Serial Chains

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Abstract. This paper proposes a formulation of inverse kinematics for serial kinematic chains defined on the hyper-state of motion, extending classical pose descriptions to include higher-order acceleration fields. The approach relies on an extension of the Lie group of rigid-body displacements using a multidual commutative nilpotent algebra. Higher-order kinematic quantities are obtained via automatic differentiation in a hypercomplex Lie algebra, thereby eliminating explicit time differentiation. The resulting kinematic mapping is expressed as a product of hypercomplex exponentials, enabling the formulation and closed-form solution of the inverse kinematics problem on the hyper-state for specific serial manipulators.

1 Introduction

In many modern robotic applications, classical inverse kinematics, formulated only at the end-effector pose level, is insufficient, since the motion must also satisfy stringent smoothness and dynamic constraints to remain safe and efficient. High-speed industrial robots, human–robot interaction systems, and medical or rehabilitation devices require a precise control of the temporal motion profile—velocity, acceleration, jerk, and higher-order derivatives—to avoid vibrations, excessive mechanical stress, or user discomfort. Previous studies on serial manipulator kinematics and minimum-jerk or minimum-snap trajectory planning have developed efficient algorithms and Lie-group formulations, but they usually treat higher-order variables as auxiliary elements rather than integrating them into a unified state space for direct inverse kinematics. This paper addresses this gap by introducing the hyper-state of motion, which combines

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the end-effector pose with its higher-order motion fields, and by formulating and solving the inverse kinematics equations in closed form on this hyper-state for specific serial 3C kinematic chains, using a hypercomplex Lie-group extension of SE (3), [1-6].

2 Hyper-state of rigid body motion

The homogenous matrix $\mathbf{g} \in \mathbf{SE}$ (3) $\mathbf{g}(\mathrm{t}) = \begin{bmatrix} \mathbf{R} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}$, $\mathrm{t} \in \mathbb{I} \subseteq \mathbb{R}$, gives the parametric equation of rigid body motion: where $\mathbf{R} = \mathbf{R}(t)$is a proper orthogonal tensor [1], [2], and $\boldsymbol{\rho} = \boldsymbol{\rho}(t)$, are vector functions of a time variable. The rotation tensor function $\mathbf{R} = \mathbf{R}(t)$ and vector function $\boldsymbol{\rho} = \boldsymbol{\rho}(t)$ are n-times differentiable $\forall \mathrm{t} \in \mathbb{I} \subseteq \mathbb{R}$. The vector fields of higher-order accelerations $\mathbf{a}_{\mathbf{r}}^{[n]}$, $\mathrm{n} \in \mathbb{N}$, of rigid body motion are presented from equations [7]:

$$\mathbf{a}_{\mathbf{r}}^{[n]} = \mathbf{a}_n + \boldsymbol{\Phi}_n, \mathbf{r} \in \mathbf{V}_3, \mathrm{n} \in \mathbb{N}. \tag{1}$$

In Eq. 1, the invariant vector $\mathbf{a}_n$ and invariant tensor $\boldsymbol{\Phi}_n$ are [7]:

$$\begin{cases} \mathbf{a}_n = \boldsymbol{\rho}^{(n)} - \boldsymbol{\Phi}_n \boldsymbol{\rho} \\ \boldsymbol{\Phi}_n = \mathbf{R}^{(n)} \mathbf{R}^T \end{cases} \tag{2}$$

We define the **hyper-state of rigid-body motion** as the set of the pose and all higher-order acceleration fields at a given instant, up to a prescribed order. Accordingly, the hyper-state may be represented by pairs consisting of vectors and Euclidean tensors:

$$\mathbf{H} = \{(\boldsymbol{\rho}, \mathbf{R}), (\mathbf{a}_1, \boldsymbol{\Phi}_1), (\mathbf{a}_2, \boldsymbol{\Phi}_2), \ldots, (\mathbf{a}_n, \boldsymbol{\Phi}_n)\} \tag{3}$$

By employing dual tensors [6], [7], [8], the hyper-state can equivalently be represented by $(n+1)$ Euclidean dual tensors:

$$\underline{\mathbf{H}} = \left\{\underline{\mathbf{R}}, \underline{\boldsymbol{\Phi}}_1, \underline{\boldsymbol{\Phi}}_2, \ldots, \underline{\boldsymbol{\Phi}}_{\mathrm{n}}\right\} \tag{4}$$

$$\underline{\mathbf{R}} = (\mathbf{I} + \varepsilon_0 \boldsymbol{\rho} \times)\mathbf{R};\ \varepsilon_0 \neq 0, \varepsilon_0^2 = 0 \tag{5}$$

$$\underline{\boldsymbol{\Phi}}_{\mathrm{k}} = \underline{\mathbf{R}}^{(k)} \underline{\mathbf{R}}^{\mathrm{T}}, k = \overline{1, n}, \tag{6}$$

In Eq. (5), $(\boldsymbol{\rho}\times)$ is the skew-symmetric Euclidean tensor corresponding to the vector $\boldsymbol{\rho}$, and $\underline{\mathbf{R}}$ is an orthogonal dual tensor isomorphic to a homogeneous matrix $\mathbf{g}=\begin{bmatrix}\mathbf{R} & \boldsymbol{\rho}\\ \mathbf{0} & 1\end{bmatrix}$.

The Lie group of rigid body displacements **SE** (3) is isomorphic to the orthogonal dual tensor set, denoted $\underline{\mathbb{SO}}_3$ [6-8].

Dual number algebra $\underline{\mathbb{R}}$, the free module of dual vectors (with operations of the scalar product $\underline{\mathbf{a}}\cdot\underline{\mathbf{b}}$, cross product $\underline{\mathbf{a}}\times\underline{\mathbf{b}}$, and triple scalar product$\langle\underline{\mathbf{a}},\underline{\mathbf{b}},\underline{\mathbf{c}}\rangle$), and dual Euclidean tensors (with operations and their invariants) were studied in [7] and [8].

The n-order multidual number algebra $\widehat{\mathbb{R}}$, the free module of dual vectors $\widehat{\mathbf{V}}_3$(by operation of scalar product, cross product, and triple scalar product), and multidual Euclidean tensors $\mathbf{L}(\widehat{\mathbf{V}}_3,\widehat{\mathbf{V}}_3)$ by operations and their invariants were studied in [6] and [10].

3 A mathematical introduction to hyper-multidual algebra

3.1 Hyper multidual (HMD) numbers and functions

Then, we will introduce the set of hyper-multidual numbers [6], [7] by tensor product of dual algebra $\underline{\mathbb{R}}$ by multidual algebra $\widehat{\mathbb{R}}$: $\underline{\widehat{\mathbb{R}}}=\underline{\mathbb{R}}\otimes\widehat{\mathbb{R}}$. $\underline{\widehat{\mathbb{R}}}$ is commutative, associative, $2(n+1)$-dimensional $\mathbb{R}$-algebra with unity. For n=1, the hyper-dual algebra is given by [8]. The generic elements of $\underline{\widehat{\mathbb{R}}}$ is:

$$\underline{\widehat{\mathbb{R}}}=\left\{\underline{\hat{x}}=\sum_{k=0}^{n}\underline{x}_k\,\varepsilon^k;\underline{x}_k\in\underline{\mathbb{R}},k=\overline{0,n}\middle|\varepsilon^j\neq 0,j=1,2,3,\ldots,n,\varepsilon^{n+1}=0\right\} \tag{7}$$

By Eq. (7), a unique decomposition for any HMD number $\underline{\hat{x}}$:

$$\underline{\hat{x}}=\hat{x}+\varepsilon_0\hat{x}_0;\hat{x},\hat{x}_0\in\widehat{\mathbb{R}} \tag{7}$$

The field of real numbers $\mathbb{R}$, dual algebra $\underline{\mathbb{R}}$, , and multidual algebra $\widehat{\mathbb{R}}$ is a subalgebras of algebra $\underline{\widehat{\mathbb{R}}}$. The zero divisors of an element $\underline{\hat{x}}\in\underline{\widehat{\mathbb{R}}}$ are characterized by $Du\underline{\hat{x}}=\underline{x}_0=0$.

We will extend the dual function of a dual variable $f:\underline{\mathbb{I}}\subseteq\underline{\mathbb{R}}\to\underline{\mathbb{R}}, f=f(\underline{x})$ to an n-times differentiable function of a hypercomplex variable $\hat{x}$ by the equation (with the notation $\underline{x}_0:=\underline{x}$):

$$f(\underline{\hat{x}})=f(\underline{x})+\sum_{k=1}^{n}\frac{(\Delta\underline{\hat{x}})^k}{k!}f^{(k)}(\underline{x}),\ Du\underline{\hat{x}}\in\ \mathbb{I}\subseteq\mathbb{R}. \tag{8}$$

In Equation (8) $\Delta\underline{\hat{x}} = \underline{\hat{x}} - \underline{x} = \sum_{k=1}^{n} \underline{x}_k \,\varepsilon^k$, is the HMD part of the variable $\underline{\hat{x}}$.

3.2 Trigonometric HMD functions and their inverse

Using Eq. (8), we obtain the following hypercomplex functions of hypercomplex variables:

$$\sin \underline{\hat{x}} = \sin \underline{x} + \sum_{k=1}^{n} \frac{(\Delta\underline{\hat{x}})^k}{k!} \sin(\underline{x} + k\frac{\pi}{2}) \,, \tag{9}$$

$$\cos \underline{\hat{x}} = \cos \underline{x} + \sum_{k=1}^{n} \frac{(\Delta\underline{\hat{x}})^k}{k!} \cos(\underline{x} + k\frac{\pi}{2}) \,, \tag{10}$$

It is proved that the system $\begin{cases} \sin \hat{x} = \frac{\hat{b}}{\sqrt{\hat{a}^2+\hat{b}^2}} \\ \cos \hat{x} = \frac{\hat{a}}{\sqrt{\hat{a}^2+\hat{b}^2}} \end{cases}$, Re $(\hat{a}^2 + \hat{b}^{2)} \neq 0$, admits a unique solution, $\hat{x} \in \hat{\mathbb{R}}$, denoted by:

$$\hat{x} = atan2(\hat{b}, \hat{a}). \tag{11}$$

For the trigonometric HMD function, if Du $(\underline{\hat{a}}^2 + \underline{\hat{b}}^2) \neq 0$:

$$atan2(\underline{\hat{b}}, \underline{\hat{a}}) = atan2(\hat{b}, \hat{a}) + \varepsilon_0 \frac{\hat{a}_0\hat{b} - \hat{a}\hat{b}_0}{\underline{\hat{a}}^2 + \underline{\hat{b}}^2} \tag{12}$$

3.3 Differential HMD transform dual time functions

Consider $\underline{f}: \mathbb{I} \subseteq \mathbb{R} \to \underline{\mathbb{R}}$, $\underline{f} = \underline{f}(t)$ a dual function of a time variable, n-times differentiable. If ε is a unit nilpotent generator of $\hat{\mathbb{R}}$, the differential operator $e^{\varepsilon \mathrm{D}} = \mathbf{I} + \varepsilon \mathrm{D} + \frac{\varepsilon^2}{2}\mathrm{D}^2 + \cdots + \frac{\varepsilon^n}{n!}\mathrm{D}^{\mathrm{n}}$ ($\mathrm{D} = \frac{d}{dt}$ the derivative operator with respect to time).

Define the differential HMD transform[10] by:

$$\underline{\check{f}} = e^{\varepsilon \mathrm{D}} \underline{f}. \tag{13}$$

Applying this transformation (Eq. 5), we obtain:

$$\underline{\hat{\mathbf{\Phi}}} = \underline{\check{\mathbf{R}}}\, \underline{\mathbf{R}}^{\mathrm{T}} = (\mathbf{I} + \varepsilon_0 \hat{\mathbf{a}} \times)\hat{\mathbf{\Phi}} \tag{14}$$

Where $(\hat{\mathbf{a}} \times)$ iis the skew-symmetric Euclidean multidual tensor corresponding to the multidual vector $\hat{\mathbf{a}}$ .From Equations (14), (15), (1), and (2), it follows that:

$$\hat{\mathbf{a}} = \mathbf{a}_1\varepsilon + \mathbf{a}_2\frac{\varepsilon^2}{2} + \cdots + \mathbf{a}_n\frac{\varepsilon^n}{n!} \tag{15}$$

$$\hat{\underline{\mathbf{\Phi}}} = \mathbf{I} + \mathbf{\Phi}_1\varepsilon + \cdots + \mathbf{\Phi}_n\frac{\varepsilon^n}{n!} \tag{16}$$

Thus, the orthogonal HMD tensor $\hat{\underline{\mathbf{\Phi}}}$ encapsulates the entire information regarding the higher-order acceleration fields of a rigid solid body.

4 Inverse Kinematics Equations on the Hyper-State of 3C Serial Kinematic Chain

Consider a spatial kinematic chain of the rigid bodies $C_k, k = \overline{0,3}$. The relative motion of the rigid body C_k with respect to the reference frame attached to C_{k-1} it, is described by the orthogonal dual tensor ${}^{k-1}\underline{\mathbf{R}}_k$. The kinematic equation of a spatial kinematic chain is [6-7]:

$$\underline{\mathbf{R}} = \exp(\underline{\alpha}_1{}^0\underline{\mathbf{u}}_1\times)\exp(\underline{\alpha}_2{}^1\underline{\mathbf{u}}_2\times)\exp(\underline{\alpha}_m{}^2\underline{\mathbf{u}}_3\times). \tag{17}$$

If unit dual vectors ${}^{k-1}\underline{\mathbf{u}}_k = \text{const.}$, $k = \overline{1,3}$, the spatial kinematic chain is named general lower-pair spatial general manipulator. The dual angles $\underline{\alpha}_k$, $k = \overline{1,3}$:

$$\underline{\alpha}_k = \alpha_k(t) + \varepsilon_0 d_k(t), t \in \mathbb{I} \subseteq \mathbb{R} \tag{18}$$

denotes the relative motion of the rigid body C_k with respect to the reference frame attached to the rigid body. The hyper-state on the terminal body of the general 3C manipulator, given by the kinematic dual mapping (32) it results from the HMD orthogonal tensor [6]:

$$\hat{\underline{\mathbf{\Phi}}} = \breve{\underline{\mathbf{R}}}\,\underline{\mathbf{R}}^T = \exp\left[(\underline{\mathbf{u}}_1\times)\Delta\breve{\underline{\alpha}}_1\right]\exp\left[(\underline{\mathbf{u}}_2\times)\Delta\breve{\underline{\alpha}}_2\right]\exp\left[(\underline{\mathbf{u}}_3\times)\Delta\breve{\underline{\alpha}}_3\right], \tag{19}$$

where $\underline{\mathbf{u}}_1 = {}^0\underline{\mathbf{u}}_1$, $\underline{\mathbf{u}}_2 = {}^0\underline{\mathbf{R}}_1{}^1\underline{\mathbf{u}}_2$, and $\underline{\mathbf{u}}_3 = {}^0\underline{\mathbf{R}}_1{}^1\underline{\mathbf{R}}_2{}^2\underline{\mathbf{u}}_3$ are unit dual vectors corresponding to the screw joint k, and $\Delta\breve{\underline{\alpha}}_k$ denote the multidual part of the hyper-multidual (HMD) variable $\breve{\underline{\alpha}}_k, k = \overline{1,m}$.

Formulation of inverse kinematics equations on the hyper-state of 3C serial chain is given by HMD equation:

$${}^0\underline{\mathbf{R}}_1\left(\breve{\underline{\alpha}}_1{}^0\underline{\mathbf{u}}_1\right){}^1\underline{\mathbf{R}}_2\left(\breve{\underline{\alpha}}_2{}^1\underline{\mathbf{u}}_2\right){}^2\underline{\mathbf{R}}_3\left(\breve{\underline{\alpha}}_3{}^2\underline{\mathbf{u}}_3\right) = \breve{\underline{\mathbf{R}}} \tag{20}$$

In Eq. (15), the HMD orthogonal tensor $\underline{\breve{\mathbf{R}}} = \widehat{\mathbf{\Phi}}\mathbf{R}$ and the unit dual vectors, ${}^{k-1}\underline{\mathbf{u}}_k$, $k = \overline{1,\ 3}$, are assumed to be known, and $\underline{\breve{\alpha}}_k, = \overline{1,\ 3}$, are to be determined.

Equation (21) admits a closed-form solution.

Let the HMD orthogonal tensor (matrix) $\underline{\breve{\mathbf{R}}} \in \underline{\widehat{\mathbb{SO}}_3}$, and ${}^0\underline{\mathbf{u}}_1$, ${}^1\underline{\mathbf{u}}_2$, ${}^2\underline{\mathbf{u}}_3$ three-unit dual vectors, with $\langle {}^0\underline{\mathbf{u}}_1, {}^1\underline{\mathbf{u}}_2, {}^2\underline{\mathbf{u}}_3 \rangle \neq 0$.

We will determine the conditions under which the HMD angles $\underline{\breve{\alpha}}_1$, $\underline{\breve{\alpha}}_2$, and $\underline{\breve{\alpha}}_3$ exist so that Eq. (21) holds.

Now we can give the following theorem:

Main Theorem: *If* $\underline{\breve{\mathbf{R}}} \in \underline{\widehat{\mathbb{SO}}_3}$ *and* ${}^0\underline{\mathbf{u}}_1$, ${}^1\underline{\mathbf{u}}_2$, ${}^2\underline{\mathbf{u}}_3$ *three-unit dual vectors, so that* $Re\left({}^1\underline{\mathbf{u}}_2 \times {}^0\underline{\mathbf{u}}_1\right) \neq \mathbf{0}$, $Re\left({}^1\underline{\mathbf{u}}_2 \times {}^2\underline{\mathbf{u}}_3\right) \neq \mathbf{0}$, *the Equation (21) having unknown, the HMD angles* $\underline{\breve{\alpha}}_k$ *with* $k = \overline{1,3}$, *admits solutions if and only if* ($\mathbf{u}_k = \mathrm{Re}\left({}^{k-1}\underline{\mathbf{u}}_k\right)$, *and* $\mathbf{R} = \mathrm{Re}(\underline{\breve{\mathbf{R}}})$)

$$|\mathbf{u}_1(\mathbf{R} - \mathbf{u}_2 \otimes \mathbf{u}_2)\mathbf{u}_3| \leq \|\mathbf{u}_2 \times \mathbf{u}_1\| \|\mathbf{u}_2 \times \mathbf{u}_3\| \tag{21}$$

$$\mathbf{R}\mathbf{u}_3 \neq \pm \mathbf{u}_1 \tag{22}$$

Proof: From Eq. (21), after a double contraction [11] of the tensor $\underline{\breve{\mathbf{R}}}$ with the unit dual vectors ${}^0\underline{\mathbf{u}}_1$ and ${}^2\underline{\mathbf{u}}_3$, we obtain:

$${}^0\underline{\mathbf{u}}_1\, {}^1\underline{\mathbf{R}}_2\left(\underline{\breve{\alpha}}_2\, {}^1\underline{\mathbf{u}}_2\right) {}^2\underline{\mathbf{u}}_3 = {}^0\underline{\mathbf{u}}_1 \underline{\breve{\mathbf{R}}}\, {}^2\underline{\mathbf{u}}_3 \tag{23}$$

After calculations, $\breve{\alpha}_2$we obtain the equation:

$$\underline{\widehat{a}}\, cos\underline{\breve{\alpha}}_2 + \underline{\widehat{b}}\, sin\underline{\breve{\alpha}}_2 = \underline{\widehat{c}} \tag{24}$$

with the notations:

$$\underline{\widehat{a}} = \left({}^1\underline{\mathbf{u}}_2 \times {}^0\underline{\mathbf{u}}_1\right) \cdot \left({}^1\underline{\mathbf{u}}_2 \times {}^2\underline{\mathbf{u}}_3\right) \tag{25}$$

$$\underline{\widehat{b}} = \langle {}^0\underline{\mathbf{u}}_1, {}^1\underline{\mathbf{u}}_2, {}^2\underline{\mathbf{u}}_3 \rangle \tag{26}$$

$$\underline{\widehat{c}} = {}^0\underline{\mathbf{u}}_1 \left[\, \underline{\breve{\mathbf{R}}} - {}^1\underline{\mathbf{u}}_2 \otimes {}^1\underline{\mathbf{u}}_2 \right] {}^2\underline{\mathbf{u}}_3 \tag{27}$$

In Eq. (28) ${}^1\underline{\mathbf{u}}_2 \otimes {}^1\underline{\mathbf{u}}_2$ denote dyadic product of dual vectors. Under the conditions given by the Main Theorem, the solution is obtained:

$$\boxed{\underline{\breve{\alpha}}_2 = \mathrm{atan2}\left[\underline{\widehat{b}\widehat{c}} \pm \underline{\widehat{a}\widehat{d}},\ \underline{\widehat{a}\widehat{c}} \mp \underline{\widehat{b}\widehat{d}}\right]} \tag{28}$$

with $\underline{\widehat{d}}$ is given by:

$$\underline{\hat{\mathrm{d}}} = \sqrt{\left\|{}^1\underline{\mathbf{u}}_2 \times {}^0\underline{\mathbf{u}}_1\right\|^2 \left\|{}^1\underline{\mathbf{u}}_2 \times {}^2\underline{\mathbf{u}}_3\right\|^2 - \left[{}^0\underline{\mathbf{u}}_1\left[\,\underline{\breve{\mathbf{R}}} - {}^1\underline{\mathbf{u}}_2 \otimes {}^1\underline{\mathbf{u}}_2\right]{}^2\underline{\mathbf{u}}_3\right]^2} \tag{29}$$

The hyper-multidual angles $\underline{\breve{\alpha}}_1$and $\underline{\breve{\alpha}}_3$are obtained similarly to $\underline{\breve{\alpha}}_2$, using the identities:

$${}^1\underline{\mathbf{R}}_2\left(\underline{\breve{\alpha}}_2{}^1\underline{\mathbf{u}}_2\right){}^0\underline{\mathbf{R}}_1\left(\underline{\breve{\alpha}}_1\underline{\hat{\mathbf{v}}}_3\right){}^2\underline{\mathbf{R}}_3\left(\underline{\breve{\alpha}}_3{}^2\underline{\mathbf{u}}_3\right) = {}^0\underline{\mathbf{R}}_1\left(\underline{\breve{\alpha}}_1{}^0\underline{\mathbf{u}}_1\right){}^2\underline{\mathbf{R}}_3\left(\underline{\breve{\alpha}}_3\underline{\hat{\mathbf{v}}}_1\right){}^1\underline{\mathbf{R}}_2\left(\underline{\breve{\alpha}}_2{}^1\underline{\mathbf{u}}_2\right) = \underline{\breve{\mathbf{R}}} \tag{30}$$

$$\underline{\widehat{\mathbf{R}}}_2 = \exp(\underline{\breve{\alpha}}_2{}^1\underline{\mathbf{u}}_2\,\times) \tag{31}$$

$$\underline{\hat{\mathbf{v}}}_1 = \underline{\widehat{\mathbf{R}}}_2{}^2\underline{\mathbf{u}}_3 \tag{32}$$

$$\underline{\widehat{\mathbf{w}}}_1 = \underline{\breve{\mathbf{R}}}\,{}^2\underline{\mathbf{u}}_3 \tag{33}$$

$$\boxed{\underline{\breve{\alpha}}_1 = \operatorname{atan2}\left[\langle{}^0\underline{\mathbf{u}}_1, \underline{\hat{\mathbf{v}}}_1, \underline{\widehat{\mathbf{w}}}_1\rangle,\ \left({}^0\underline{\mathbf{u}}_1 \times \underline{\hat{\mathbf{v}}}_1\right)\cdot\left({}^0\underline{\mathbf{u}}_1 \times \underline{\widehat{\mathbf{w}}}_1\right)\right]} \tag{34}$$

$$\underline{\hat{\mathbf{v}}}_3 = \underline{\widehat{\mathbf{R}}}_2^T{}^0\underline{\mathbf{u}}_1 \tag{35}$$

$$\underline{\widehat{\mathbf{w}}}_3 = \underline{\breve{\mathbf{R}}}^T\mathbf{u}_1 \tag{36}$$

$$\boxed{\underline{\breve{\alpha}}_3 = \operatorname{atan2}\left[\langle{}^2\underline{\mathbf{u}}_3, \underline{\widehat{\mathbf{w}}}_3,\ \underline{\hat{\mathbf{v}}}_3\rangle,\ \left({}^2\underline{\mathbf{u}}_3 \times \underline{\hat{\mathbf{v}}}_3\right)\cdot\left({}^2\underline{\mathbf{u}}_3 \times \underline{\widehat{\mathbf{w}}}_3\right)\right]} \tag{37}$$

In equations (29), (35), and (38), we have:

$$\underline{\breve{\alpha}}_\mathrm{k} = \underline{\alpha}_k + \underline{\dot{\alpha}}_k\varepsilon + \underline{\ddot{\alpha}}_k\frac{\varepsilon^2}{2} + \cdots. + \alpha_k^{(n)}\frac{\varepsilon^n}{n!};\mathrm{k} = 1,2,3. \tag{38}$$

5 Computation Algorithm and Numerical Example

A representative proper orthogonal HMD target tensor was considered, and Algorithm 1 was applied to recover the corresponding joint hyper-angles.

Algorithm 1. Closed-Form Inverse Kinematics on the Hyper-State.

Input data: $\underline{\breve{\mathbf{R}}}$, ${}^0\underline{\mathbf{u}}_1$, ${}^1\underline{\mathbf{u}}_2$, ${}^2\underline{\mathbf{u}}_3$

1. ***Verify the Main Theorem constraints*** *using (22)–(23).*
2. ***Compute*** *the intermediate HMD scalars* $\underline{\hat{\mathrm{a}}}, \underline{\hat{\mathrm{b}}}, \underline{\hat{\mathrm{c}}}, \underline{\hat{\mathrm{d}}}$, *using (25)–(28) and (30).*
3. ***Evaluate*** *the two solutions for* $\underline{\breve{\alpha}}_2$ *using (29).*
4. *For each solution* $\underline{\breve{\alpha}}_2$*:* ***construct*** $\underline{\widehat{\mathbf{R}}}_2$ *using (32),* ***construct*** $\underline{\hat{\mathbf{v}}}_1$ *and* $\underline{\widehat{\mathbf{w}}}_1$ *using (33) and (34),* ***compute*** $\underline{\breve{\alpha}}_1$ *using (35),* ***form*** $\underline{\hat{\mathbf{v}}}_3$ *and* $\underline{\widehat{\mathbf{w}}}_3$ *using (36),(37),* ***compute*** $\underline{\breve{\alpha}}_3$ *using(38).*

Output data: - solutions for $\underline{\breve{\alpha}}_1, \underline{\breve{\alpha}}_2, \underline{\breve{\alpha}}_3$

Below, a numerical example illustrating the extraction of the angles $\underline{\breve{\alpha}}_1$, $\underline{\breve{\alpha}}_2$ and $\underline{\breve{\alpha}}_3$ using Algorithm 1 implemented in MATLAB R2025b is presented. All theorem-side conditions were satisfied, and the reconstructed product matched the prescribed tensor up to numerical precision. In MATLAB, in addition to the extracted values, the orthogonality conditions and the conditions of **Main Theorem** were also verified. All theorem-side conditions were satisfied, and the reconstructed product matched the prescribed tensor up to numerical precision.

```
--------------------------------INPUT DATA----------------------------
 u1_ =        u2_ =                                u3_ =
 [1 ]         [(0.707106781)+eps0*(0.353553391)]   [(-0.707106781)+eps0*(1) ]
 [0 ]         [0 ]                                 [(-0.707106781)+eps0*(-1)]
 [eps0*(1)]   [(0.707106781)+eps0*(0.353553391)]   [0 ]
R_target_hat_=
   columns 1:1
              [(1) + eps0*(-6*eps^3) ]
              [(3*eps^3) + eps0*(19*eps^2) ]
              [(-1*eps^2) + eps0*(-6*eps + 38*eps^3)]
   columns 2:2
              [(3*eps^3) + eps0*(18*eps^2) ]
              [(1 - 18*eps^2) + eps0*(-6*eps + 30*eps^3) ]
              [(6*eps - 36*eps^3) + eps0*(1 - 17*eps^2 + 1*eps^3)]
   columns 3:3
              [(1*eps^2) + eps0*(6*eps - 35*eps^3) ]
              [(-6*eps + 36*eps^3) + eps0*(-1 + 17*eps^2 - 1*eps^3)]
              [(1 - 18*eps^2) + eps0*(-6*eps + 24*eps^3) ]
```

Fig. 1. MATLAB input data for algorithm 1.

```
------------------------ PRIMARY SOLUTION ---------------------------
alpha1_hat_ = (3.141592654 + 6*eps - 1*eps^2 - 3*eps^3) + eps0*(-1 -
12*eps + 4.828427125*eps^2 + 54*eps^3)
alpha2_hat_ = (-1.910633236 + 4.242640687*eps^3) + eps0*(-3.138071187
+ 8.485281374*eps + 24.04163056*eps^2 - 53.033008589*eps^3)
alpha3_hat_ = (1.910633236 - 1.414213562*eps^2) + eps0*(-5.1045695 -
8.485281374*eps + 27.455844123*eps^2 + 32.526911935*eps^3)
------------------------ SECONDARY SOLUTION -------------------------
alpha1_hat_ = (6*eps - 1*eps^2 + 3*eps^3) + eps0*(1 +
38.828427125*eps^2 - 6*eps^3)
alpha2_hat_ = (-4.242640687*eps^3) + eps0*(-8.485281374*eps -
24.04163056*eps^2 + 53.033008589*eps^3)
alpha3_hat_ = (-1.414213562*eps^2) + eps0*(-8.485281374*eps +
27.455844123*eps^2 + 41.012193309*eps^3)
```

Fig. 2. MATLAB output data for algorithm 1.

The obtained solutions are summarized in the following table, where the results are given directly in terms of the extracted joint angular quantities $(\alpha, \dot{\alpha}, \ddot{\alpha}, \dddot{\alpha})$ and linear quantities $\left(d, \dot{d}, \ddot{d}, \dddot{d}\right)$.

Table 1. Kinematic quantities extracted

	i	α_i	$\dot{\alpha}_i$	$\ddot{\alpha}_i$	$\dddot{\alpha}_i$	d_i	$\dot{d}_i$	$\ddot{d}_i$	$\dddot{d}_i$
		$[rad]$	$[rad/s]$	$[rad/s^2]$	$[rad/s^3]$	$[mm]$	$[mm/s]$	$[mm/s^2]$	$[mm/s^3]$
Solution 1	1	3.1415	6.0000	-2.0000	-18.0000	-1.0000	-12.0000	9.6568	324.0000
	2	-1.9106	0.0000	0.0000	25.4558	-3.1380	8.4852	48.0832	-318.1980
	3	1.9106	0.0000	-2.8284	0.0000	-5.1045	-8.4852	54.9116	195.1614
Solution 2	1	0.0000	6.0000	-2.0000	18.0000	1.0000	0.0000	77.6568	-36.0000
	2	0.0000	0.0000	0.0000	-25.4558	0.0000	-8.4852	-48.0832	318.1980
	3	0.0000	0.0000	-2.8284	0.0000	0.0000	-8.4852	54.9116	246.0731

6 Conclusions

This paper formulated the inverse kinematics problem for serial kinematic chains in the extended hyper-state space, incorporating pose and higher-order acceleration in a unified framework. Using hypercomplex homogeneous transformations, the inverse problem was expressed as a hypercomplex exponential product equation with joint hyper-variables as unknowns. For six-degree-of-freedom serial manipulators, existence conditions and closed-form solutions were derived. The proposed framework provides a compact, efficient foundation for higher-order inverse kinematics and supports future developments in real-time robotic motion generation.

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