Balance Solutions for the Chebyshev Lambda Straight-Line Linkage

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Abstract This paper presents the general balance solutions and the special balance solutions of the Chebyshev Lambda Straight-Line Linkage. The general balance solutions are related to those of the general 4R Four-Bar linkage, of which one solution which in general is considered disadvantageous in the design turns out advantageous here due to the extended coupler link. The special balance solutions utilize the straight-line motion path which, when oriented horizontally, results in balance solutions with only one countermass instead of the usual two. The interesting application of the special balance solutions as constant force generators with the straight-line motion path oriented vertically or diagonally is shown and explained.

1 Introduction

For a six meter high sculpture of stainless steel that collects rainwater for an interactive use by the spectator, I developed a sweeper mechanism based on the Chebyshev Lambda Straight-Line Linkage for sweeping water from a horizontal surface. Since the spectator needs to move the sweeper by hand, this motion needs to be effortless no matter its weight. Also for safety the mechanism must not start moving by itself due to gravity, as this leads to an increased risk of serious injury when a hand or finger might be stuck in between the moving elements. Both objectives can be achieved by a force balanced design of the linkage. Here force balance refers to both gravity force balance and shaking force balance by the mass distribution of the linkage [**?**].

Since for elegance of the sculpture I allowed only one countermass to be placed on the crank link although also a countermass on the rocker link is

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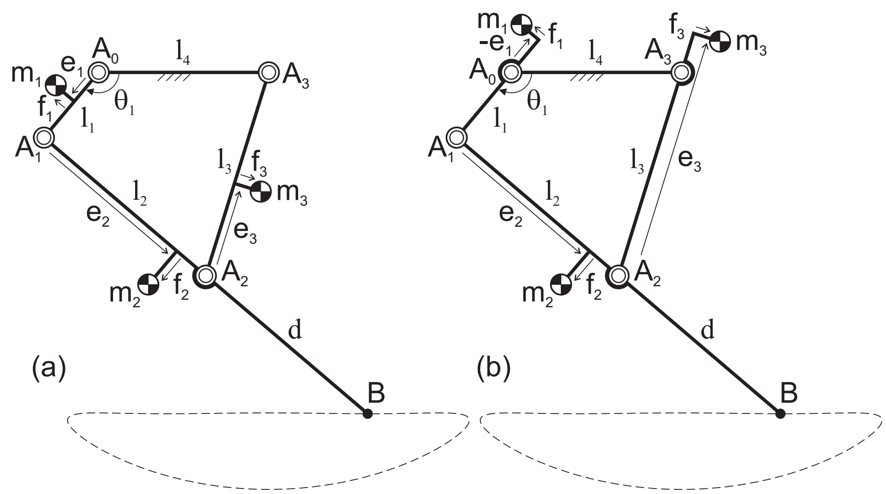

Fig. 1 (a) Chebyshev λ Straight-Line Linkage and (b) its general balance solution.

required for balancing a general 4R four-bar linkage, I was experimentally testing in a prototype setup how well I could approximately balance the linkage. To my surprise the balance turned out so well that I assumed it must be a perfect solution. From this observation I learned that also for the Chebyshev Lambda Straight-Line Linkage special balance solutions exist that can be derived in a comparable way as those found for the Peaucellier-Lipkin Straight-Line Linkage which I published earlier [**?**]. This can be explained as that part of the linkage mass is modelled in the link point tracing the straight line such that for the straight line being horizontal (i.e. orthogonal to the direction of gravity) this part of the mass no longer needs to be balanced in the vertical direction for motion along the straight line.

In this paper the general and the special balance solutions of the Chebyshev Lambda Straight-Line Linkage are derived and presented, which can be useful for various other applications of the linkage such as in robotic walking legs [**?**] and Chebyshev's plantigrade machines, vehicle suspensions [**?**], and surgical instruments [**?**].

2 General balance solutions for full motion cycle

Figure 1a shows the Chebyshev Lambda Straight-Line Linkage, a 4R four-bar linkage $A_0A_1A_2A_3$ with base joints A_0 and A_3, a crank with length l_1, a coupler link with length l_2 and a rocker link with length l_3. The coupler link is extended with a collinear length d with endpoint B which traces an approximate straight line for a significant part of its full curve when $l_1 : l_2 : l_3 : l_4 : d = 1 : 2.5 : 2.5 : 2 : 2.5$.

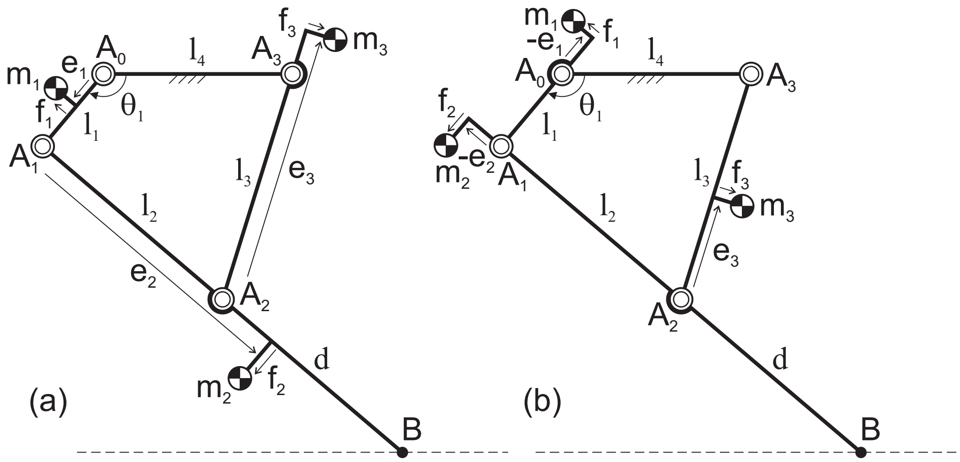

Fig. 2 Balance solutions based on (a) the crank and (b) the rocker.

Each link has a mass m_i of which the center of mass (CoM) is defined with parameters e_i and f_i, as illustrated. The general balance conditions for which the common CoM of the three links is in a stationary point in the base for all motions are the same as for the 4R four-bar linkage and write [**?**, **?**]:

$$\begin{array}{ll} m_1e_1 + m_2(1 - \frac{e_2}{l_2})l_1 = 0 & m_1f_1 - m_2\frac{f_2}{l_2}l_1 = 0 \\ m_3(l_3 - e_3) + m_2\frac{e_2}{l_2}l_3 = 0 & m_3f_3 - m_2\frac{f_2}{l_2}l_3 = 0 \end{array} \tag{1}$$

Figure 1b shows the most common balance solution which is based on the coupler link with the crank and the rocker links having countermasses such that their link CoMs are located at the opposite site of the base joints.

Figure 2 shows two other implementations of the balance conditions. Figure 2a shows the solution that is based on the crank resulting in the CoM of the coupler to be on the other side of A_2 as illustrated. Generally this is a less favorable design requiring a countermass on the coupler link leading to a much higher countermass about the base joint and a high total mass of the linkage. However for the Chebyshev Lambda Linkage it is likely advantageous due to the mass of link section A_2B and perhaps an end-effector mass in B to be included for force balance, for which no additional countermass on the coupler link is needed. This means that the balance solution would require solely one countermass, about A_3, instead of the usual two.

Figure 2b shows a solution similar to Fig. 2a but based on the rocker instead of the crank, resulting in the CoM of the coupler to be on the other side of A_1 as illustrated. This solution is also for the Chebyshev Lambda Linkage less favorable as it requires a countermass on the coupler link.

For mass symmetric links, i.e. links of which the CoM is on the line through its joints with $f_1 = f_2 = f_3 = 0$, the balance solutions are reduced. The

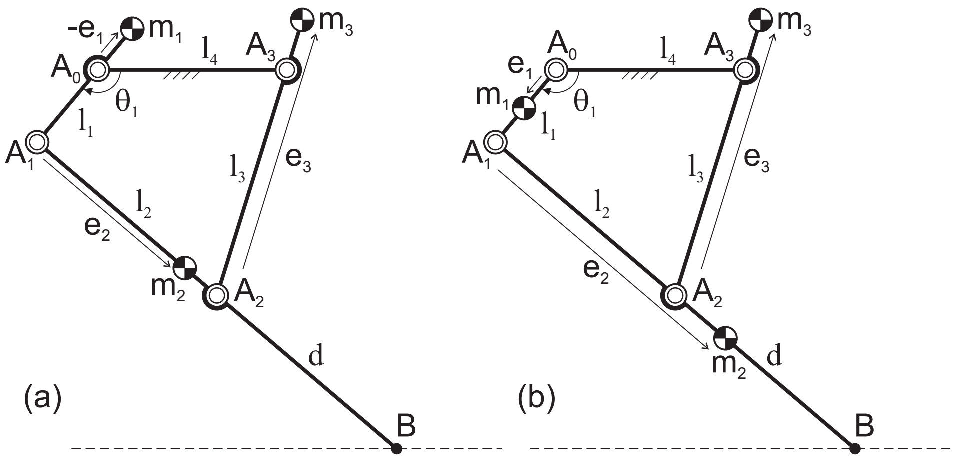

Fig. 3 Balance solutions of (a) Fig. 1b and (b) Fig. 2a for mass symmetric links.

solutions in Figs. 1b and 2a then become as illustrated in Fig. 3a and 3b, respectively. From the balance conditions (1) the locations of link CoMs 1 and 3 in Fig. 3a can be derived as:

$$e_1 = -\tfrac{m_2}{m_1}(1 - \tfrac{e_2}{l_2})l_1 \qquad e_3 = \tfrac{m_2 e_2}{m_3 l_2} l_3 + l_3 \tag{2}$$

and the locations of link CoMs 2 and 3 in Fig. 3b can be derived as:

$$e_2 = \tfrac{m_1 e_1}{m_2 l_1} l_2 + l_2 \qquad e_3 = \tfrac{m_2 e_2}{m_3 l_2} l_3 + l_3 \tag{3}$$

3 Special balance solutions for straight-line motion

The solutions in Figs. 1, 2, and 3 are perfectly balanced for the complete motion cycle, for both the straight and the curved part of the coupler curve of B. Also these balanced linkages can have any orientation, e.g. with the straight line directed vertically. In fact the shape of the coupler curve is not relevant for their balance. However, when the straight section of the coupler curve is oriented horizontally (i.e. perpendicular to the direction of gravity), then the Chebyshev Lambda Linkage with mass symmetric links can be gravity balanced and force balanced in vertical direction for solely the straight-line section in a special way.

Similar to the Peaucellier-Lipkin straight-line linkage in [**?**], balance of the Chebyshev Lambda Linkage can be obtained by modelling an equivalent mass in the straight-line coupler point B which, due to it's horizontal motion, does not affect the balance in vertical direction and can be eliminated from the balance conditions.

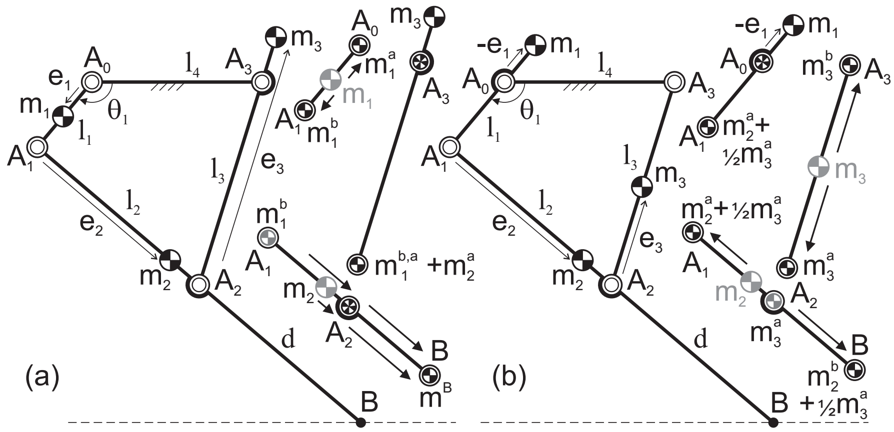

Fig. 4 Balance solutions for solely horizontal straight-line motion of B with an equivalent mass modeled in B based on (a) the crank and (b) the rocker.

Figure 4a shows the solution that has solely a counter-mass about base pivot A_3. The balance condition for this solution can be easily derived from the mass equivalent models of the links. The crank mass m_1 can be modelled equivalently with an equivalent mass $m_1^a = m_1(1-\frac{e_1}{l_1})$ in A_0 and an equivalent mass $m_1^b = m_1\frac{e_1}{l_1}$ in A_1 of which the sum is equal to m_1 and their CoM is equal to the link CoM. Similarly coupler mass m_2 can be modelled equivalently with an equivalent mass $m_2^a = m_2(1-\frac{e_2-l_2}{d})$ in A_2 and an equivalent mass $m_2^b = m_2\frac{e_2-l_2}{d}$ in B of which the sum is equal to m_2 and their CoM is equal to the link CoM. Also the equivalent mass m_1^b in A_1 can be modelled equivalently with an equivalent mass $m_1^{b,a} = m_1^b(1-\frac{-l_2}{d})$ in A_2 and an equivalent mass $m_1^{b,b} = m_1^b\frac{-l_2}{d}$ in B.

As a result the mass model of the rocker link 3 ends up with an equivalent mass of $(m_1^b(1-\frac{-l_2}{d}) + m_2^a)$ in A_2. Since for balance the CoM of this link model must be in A_3, the balance condition for the linkage in Fig. 4a is derived as:

$$m_3(l_3 - e_3) + (m_1^b(1-\frac{-l_2}{d}) + m_2^a)l_3 = 0 \tag{4}$$

and the total equivalent mass modelled in B is calculated as $m^B = m_1^b\frac{-l_2}{d} + m_2^b$. As compared to the balance solution in Fig. 3a here the crank does not need a countermass and as compared to the balance solution in Fig. 3b here the coupler mass does not need to act as a countermass and can be anywhere along the link.

Figure 4b shows a second solution that has solely a counter-mass about base pivot A_0. The balance condition for this solution can also be easily derived from the mass equivalent models of the links. The rocker mass m_3 can be modelled equivalently with an equivalent mass $m_3^a = m_3(1-\frac{e_3}{l_3})$ in

A_2 and an equivalent mass $m_3^b = m_3 \frac{e_3}{l_3}$ in A_3 of which the sum is equal to m_3 and their CoM is equal to the link CoM. Similarly coupler mass m_2 can be modelled equivalently with an equivalent mass $m_2^a = m_2(1 - \frac{e_2}{l_2+d})$ in A_1 and an equivalent mass $m_2^b = m_2 \frac{e_2}{l_2+d}$ in B of which the sum is equal to m_2 and their CoM is equal to the link CoM. Also the equivalent mass m_3^a in A_2 can be modelled equivalently with an equivalent mass $\frac{m_3^a}{2}$ in A_1 and an equivalent mass $\frac{m_3^a}{2}$ in B since l_2 is equal to d. As a result the mass model of the crank ends up with an equivalent mass of $m_2^a + \frac{m_3^a}{2}$ in A_1. Since for balance the CoM of this link must be in A_0, the balance condition for the linkage in Fig. 4b is derived as:

$$m_1 e_1 + (m_2^a + \frac{m_3^a}{2}) l_1 = 0 \tag{5}$$

and the total equivalent mass modelled in B is calculated as $m_2^b + \frac{m_3^a}{2}$. Also for this solution there is only one countermasses required, here located on the crank, and the coupler mass does not need to act as a countermass as in Fig. 3b but can be anywhere along the link.

For both linkages in Fig. 4 the common CoM of all three links moves along a path that is similar to the coupler curve of point B but scaled. The CoM of link mass m_2 can be anywhere along the line through A_1A_2, also on other side of A_2 and any additional mass due to, for instance, an end-effector can be implemented with its CoM in B without affecting the balance. Joint masses and mass of additional components can be included in the total mass m_i of the link to which they are attached. For the joints the choice to which of the jointed links to add the mass to is free and does not affect the overal outcome. Since the straight line of the Chebyshev Lambda Linkage is an approximate straight line, also the balance obtained with the solutions in Fig. 4 is an approximate balance since there is a slight vertical motion of B. Yet each balance solution itself is exact. By optimization of the countermass parameters, deviating from the exact values, it might be possible to obtain even nearer to perfect balance performance along the straight line. For motion along the curved section of the coupler curve they remain unbalanced.

It is interesting to consider the balanced linkages in Fig. 4 in an orientation where the straight-line is oriented vertically or diagonally, as shown for the linkage of Fig. 4b in Fig. 5. Since the straight line is no longer horizontal the linkage is no longer balanced for motion along the straigth line. However, here the point B can exert a force that is constant when moving along the straight path and in any point along the straight path, making it a constant force generator. The constant force F_c is proportional to the total equivalent mass modelled in B, $m_2^b + \frac{m_3^a}{2}$, and depends on the angle α with respect to the vertical, as illustrated, and is calculated as $F_c = (m_2 \frac{e_2}{l_2+d} + \frac{m_3}{2}(1 - \frac{e_3}{l_3}))g \cos\alpha$.

The linkages in Fig. 4 can be designed such that they either generate a constant force downwards or in the opposite direction upwards. In general the solution in Fig. 4a will generate a constant force upwards since the mass

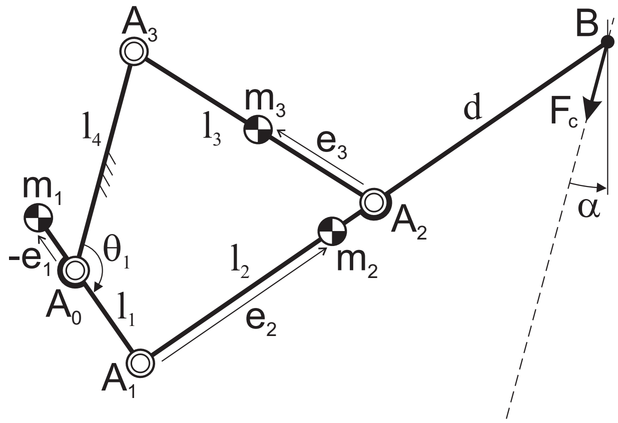

Fig. 5 Balanced Chebyshev Lambda Linkage applied as a constant force generator exerting a constant force F_c along the straight-line.

Table 1 Parameter values for a numerical example of Fig. 4b and Fig. 5

	[mm]		[mm]		[kg]		[mm]	
l_1	250	l_4	500	m_1	8.58	e_1	-117.98	$\alpha = 15°$
l_2	625	d	625	m_2	7.18	e_2	685.00	$F_c = 44.9N$
l_3	625			m_3	3.15	e_3	306.00	

modeled in B will usually be negative, while the solution in Fig. 4b and Fig. 5 will generally generate a constant force downwards since the mass modeled in B will usually be positive. If static balance for a diagonal or vertical orientation of the straight path is anyway required, then this is still possible by counteracting the force F_c with a constant force spring of equal value acting in opposite direction. In Table 1 the parameter values are presented of a realistic numerical example of the linkage in Fig. 4b and Fig. 5.

4 Discussion and Conclusion

This paper presented the general balance solutions and the special balance solutions of the Chebyshev Lambda Straight-Line Linkage. First the general balance solutions were obtained, which are related to those of the general 4R Four-Bar linkage. Here one solution which in general is considered disadvantageous in the design showed advantageous due to the extended coupler link. Subsequently the special balance solutions were obtained which utilize the horizontal straight-line motion path. Since part of the mass of the linkage could be modelled in the coupler point tracing the straight line, balance solutions were obtained that need only one countermass instead of the usual two. It has been shown how these special balance solutions can be applied as a constant force generator when the straight-line motion path is oriented vertically or diagonally and a numerical example was presented.

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