On Kinematic Acquisition of Geometric Constraints Via Null-Space Analysis of Planar Displacement Data

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Abstract This paper studies the problem of extracting geometric constraints from a set of planar displacements or pose data. A generalized pose matrix is used to capture all given pose data, and a null space analysis method is introduced to obtain a set of eight homogeneous parameters that are compatible with the given pose data. A pair of quadratic conditions on these homogeneous parameters is used to identify the underlying geometric constraints. The null-space method extends the previous algebraic fitting method of 1-DOF motions to 2- and 3-DOF motions. Three examples are provided to analyze the pose data generated from a 2-DOF RR dyad, a 1-DOF $4R$ linkage, and a 2-DOF $5R$ linkage, respectively.

1 Introduction

The prevailing approach to kinematic design of mechanisms, whether single degree of freedom (DOF) or multi-DOF robotic mechanisms, follows the paradigm of separating type synthesis and dimensional synthesis. For the latter, exact dimensional synthesis techniques have been extensively developed for a limited number of prescribed task poses [10, 13, 14]. Since early work on type synthesis of planar linkages [1, 16], there has been significant interest and progress, including Yan [18] in the context of creative design of mechanical devices and Mruthyunjaya [12] on a comprehensive review of

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existing methods. In recent decades, there has been a rapidly growing interest in extending type synthesis from single DOF mechanisms to multi-DOF robotic mechanisms. Major works in this area include seminal methodologies by Hervé [6] using group mathematics and Gogu [5] on structural synthesis, alongside Tsai [15] on enumeration of kinematic structures according to function, Huang [7] on screw theory based methods, Kong and Gosselin [8] on a comprehensive treatment, as well as Gao et al. [2] and Meng et al. [11] on the development of GF set theory for type synthesis addressing challenges in combining rotational and translational DOFs.

This paper follows our recent work on a unified formulation for the algebraic fitting of planar four-bar linkages [4, 3, 19, 9] and focuses on the problem of extracting geometric constraints from a set of pose data of a planar object. This problem has been termed "kinematic acquisition of geometric constraints" in [17], where simple geometric constraints were enumerated and studied. In the present paper, we seek to bring these two threads of research together and present a null-space analysis method for extracting not only dimensions but also the number of geometric constraints compatible with the given dataset. While this paper focuses on planar linkages with revolute joints, the approach seems to be promising in leading towards a methodology that extracts not only link dimensions and mechanism types, but also number of DOFs from a given dataset, without prior knowledge of the underlying geometric constraints.

The organization of the paper is as follows. Section 2 reviews the kinematic preliminaries on point transformations. Section 3 formulates a quasi-linear representation for planar dyads and a generalized pose matrix constructed from all given poses. Section 4 presents a null-space approach for acquiring geometric constraints. Section 5 provides three examples corresponding to planar motions under one circle constraint, two circle constraints, and one circle constraint with one circular ring constraint.

2 Kinematics Preliminaries

Consider a planar displacement $\mathbf{M}(d_x, d_y, \theta)$, where (d_x, d_y) and θ represent the location and orientation of the moving frame M relative to the fixed frame F, respectively (Fig. 1). The displacements of a moving link in a planar mechanism are subject to geometric constraints imposed by the kinematic structure of the mechanism. As points and lines are the basic elements of plane geometry, these constraints can be expressed as equations on their coordinates. While this paper restricts its focus to point transformations to illustrate the framework for kinematic acquisition of geometric constraints, the methodology is fundamentally scalable to line coordinates.

Let $\mathbf{x} = (x_1, x_2, 1)$ and $\mathbf{X} = (X_1, X_2, 1)$ denote the homogeneous coordinates of a point measured in M and F, respectively. They are related by

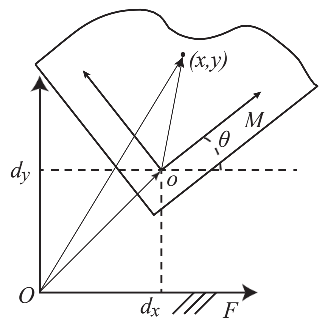


Fig. 1: A planar displacement or pose $\mathbf{M}(d_x, d_y, \theta)$.

$\mathbf{X} = [H]\mathbf{x}$, where

$$[H] = \begin{bmatrix} \cos\theta & -\sin\theta & d_x \\ \sin\theta & \cos\theta & d_y \\ 0 & 0 & 1 \end{bmatrix}. \quad (1)$$

A planar motion for which one of its moving points, $\mathbf{X}$, stays on a circle can be expressed as $(X_1 - a_1)^2 + (X_2 - a_2)^2 = r^2$, where $\mathbf{a} = (a_1, a_2)$ is the center of the circle and r is its radius. Ge et al. [3] recast it into

$$2a_1X_1 + 2a_2X_2 + a_3 = a_0(X_1^2 + X_2^2), \quad (2)$$

where $a_3 = r^2 - a_1^2 - a_2^2$, so that it can represent either a circle ($a_0 = 1$) or a line ($a_0 = 0$). A circle constraint yields a RR dyad, whereas a linear constraint produces a PR dyad, where R and P denote revolute and prismatic joints, respectively.

3 Geometrically Constrained Planar Motions

Using a planar quaternion formulation, Ge et al. [3] shows that by substituting $\mathbf{X} = [H]\mathbf{x}$ into Eq. (2), the geometric constraints associated with RR and PR dyads can be expressed in a quasi-linear representation, which can be rewritten using point transformation matrix in Eq. (1) directly as

$$\frac{d_x^2 + d_y^2}{4}p_1 + \frac{\bar{d}_x}{2}p_2 + \frac{\bar{d}_y}{2}p_3 + \frac{d_x}{2}p_4 + \frac{d_y}{2}p_5 + \frac{\sin\theta}{2}p_6 + (-\cos\theta)p_7 + p_8 = 0, \quad (3)$$

with $\bar{d}_x = -d_x\cos\theta - d_y\sin\theta$, $\bar{d}_y = d_x\sin\theta - d_y\cos\theta$. Vector $\mathbf{p} = (p_1, p_2, \cdots, p_8)^T$, a set of eight homogeneous parameters, must satisfy the following two quadratic constraints:

$$Q_1(\mathbf{p}) = p_1p_6 + p_2p_5 - p_3p_4 = 0, \quad Q_2(\mathbf{p}) = 2p_1p_7 - p_2p_4 - p_3p_5 = 0. \quad (4)$$

The focus of the present paper is on the circular constraint only, i.e., RR dyad. In this case, the location of the moving point (x_1, x_2) in M and the circle center (a_1, a_2) in F can be retrieved using the following:

$$x_1 = \frac{p_5p_6 - 2p_4p_7}{p_4^2 + p_5^2}, \quad x_2 = \frac{-p_4p_6 - 2p_5p_7}{p_4^2 + p_5^2}, \quad (5)$$

$$a_1 = \frac{-p_1p_4 - p_3p_6 - 2p_2p_7}{p_1^2 + p_2^2 + p_3^2}, \quad a_2 = \frac{-p_1p_5 + p_2p_6 - 2p_3p_7}{p_1^2 + p_2^2 + p_3^2}. \quad (6)$$

The radius of the circle, r, is given by

$$r^2 = a_1^2 + a_2^2 + a_3, \quad (7)$$

where

$$a_3 = \frac{(-4p_1p_8 + p_2^2 + p_3^2)(p_4^2 + p_5^2 + p_6^2 + 4p_7^2)}{(p_1^2 + p_2^2 + p_3^2)(p_4^2 + p_5^2)}. \quad (8)$$

The resulting set of five parameters, (x_1, x_2, a_1, a_2, r), defines a RR dyad.

For a given set of n planar displacements, Eq. (3) can be used to assemble a linear system $[A]\mathbf{p} = \mathbf{0}$, where $[A]$ is an $n \times 8$ matrix defined by displacement data. Each planar displacement $\mathbf{M}_i(d_{ix}, d_{iy}, \theta_i)(i = 1, 2, ..., n)$ generates a row vector $[A_{i,:}] \in \mathbb{R}^8$ of matrix $[A]$ as follows:

$$[A_{i,:}] = [A_{i1}, A_{i2}, A_{i3}, A_{i4}, A_{i5}, A_{i6}, A_{i7}, A_{i8}], \quad (9)$$

where

$$\begin{aligned} A_{i1} &= \frac{d_{ix}^2 + d_{iy}^2}{4}, \quad A_{i2} = \frac{\bar{d}_{ix}}{2}, \quad A_{i3} = \frac{\bar{d}_{iy}}{2}, \\ A_{i4} &= \frac{d_{ix}}{2}, \quad A_{i5} = \frac{d_{iy}}{2}, \quad A_{i6} = \frac{\sin\theta_i}{2}, \quad A_{i7} = -\cos\theta_i, \quad A_{i8} = 1. \end{aligned} \quad (10)$$

As the matrix $[A]$ contains all elements of the displacement or pose data, it is called the *Generalized Pose Matrix* or *G-pose matrix* in short in this paper.

4 A Null-Space Approach to Kinematic Acquisition

For a G-pose matrix $[A]$ constructed from a dataset of n poses, the linear system $[A]\mathbf{p} = \mathbf{0}$ is typically solved using the method of least squares, i.e., by minimizing $\mathbf{p}^T[A]^T[A]\mathbf{p}$. As with Ge et al. [3, 4], we use the Singular Value Decomposition (SVD) method to find the solution, i.e., $[A] = [U][S][V]^T$, where the $n \times 8$ rectangular diagonal matrix $[S]$ contains singular values and the matrix $[V]$ contains the right singular vectors. The right singular vectors associated with zero singular values define the null space of $[A]$. In this paper, when the term "singular vector" is used, it means the right singular vector only.

Here, the rank of $[A]$ plays a significant role. As a dyad is uniquely determined by five parameters, it follows that five poses are generally needed to obtain five independent equations to solve the five unknowns. This is the well-known five position synthesis problem in planar four-bar linkage design. In this case, the rank of $[A]$ is five, which means that there exist three zero singular values and the dimension of its null space is three. Ge et al. [3] presented a method to find vectors in this null space that satisfy the relations (4). The resulting vectors are then used to recover the dyad parameters using Eqs. (5) through (8).

The focus of this paper is on datasets that involve many more planar poses than five without the prior knowledge of whether these poses are compatible with a geometric constraint. In this case, there may be a need to expand the null space to an approximate null space that includes singular vectors corresponding to non-zero but small singular values. In particular, our formulation centers on the circle constraint, which corresponds to a *RR* dyad.

Let λ_j and $\mathbf{v}_j$ $(j = 1, 2, \cdots, 8)$ denote eight singular values and their corresponding singular vectors of an $n \times 8$ G-pose matrix $[A]$ arranged so that λ_j are in ascending order with λ_1 being the smallest. As singular values are always non-negative, λ_1 can be as small as zero. Singular vectors corresponding to zero singular values define the null space of $[A]$.

As singular vectors in general do not satisfy the quadratic relations (4), one needs to find a vector $\mathbf{p}$ in the linear space spanned by the singular vectors corresponding to a subset of the smallest singular values, i.e.,

$$\mathbf{p} = \sum_{k=1}^{D} \alpha_k \mathbf{v}_k, \tag{11}$$

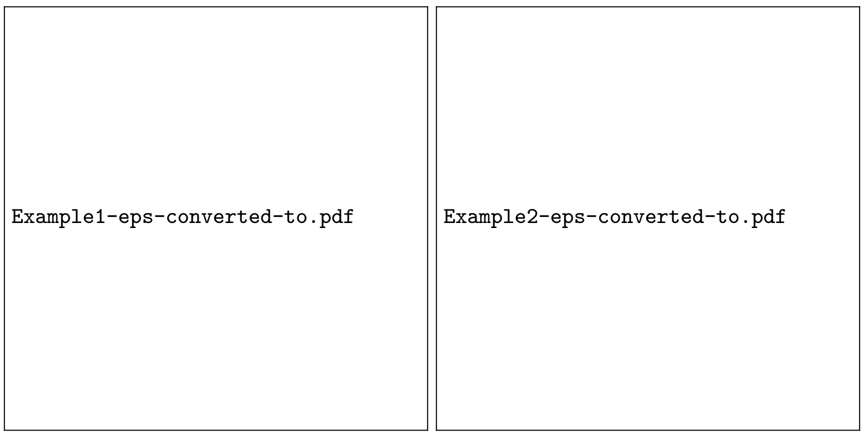
where D is the dimension of the null space or the approximate null space if one or more singular vectors of non-zero singular values are included. Substituting (11) into (4), we have

$$Q_1(\alpha_1, \cdots, \alpha_D) = 0, \quad Q_2(\alpha_1, \cdots, \alpha_D) = 0. \tag{12}$$

This leads to two equations of the second order in terms of α_k, from which one may solve for α_k, and in turn $\mathbf{p}$. As $\mathbf{p}$ is a homogeneous vector, one may normalize it such that it becomes a unit vector. This can be achieved by requiring $\sum_{k=1}^{D} \alpha_k^2 = 1$ when solving (12).

For an exact null space, this process results in a circular constraint that yields a RR dyad; for an approximate null space, it may lead to a circular ring constraint, where (a_1, a_2) defines the center of the ring, (x_1, x_2) is a variable point within the ring, and r is a variable length within the thickness of the ring. This results in a dyad with variable link length, i.e. a RPR triad. Alternatively, a RRR triad can mechanically realize this ring constraint.

5 Examples



(a) (b)

Fig. 2: (a) A set of 50 poses compatible with a RR dyad; (b) A set of 25 poses compatible with a planar $RRRR$ linkage.

Table 1: Singular vectors $\mathbf{v}$ and dyad vectors $\mathbf{p}$ for Examples 1–3

Ex.	**Vector**	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
1	$\mathbf{v_1}(\mathbf{p})$	0.0137	-0.0534	0.1166	-0.1019	-0.0029	-0.8773	0.1861	0.4107
2	$\mathbf{v_1}$	-0.0015	0.0666	0.0682	0.0800	-0.0201	-0.7290	-0.6136	-0.2761
	$\mathbf{v_2}$	0.0157	-0.0883	0.0972	-0.1474	0.0058	-0.6350	0.4782	0.5734
	$\mathbf{v_3}$	0.1029	-0.3569	0.5050	-0.7477	0.0795	0.0896	-0.1323	-0.1269
	$\mathbf{p_1}$	0.0137	-0.0534	0.1166	-0.1019	-0.0029	-0.8773	0.1861	0.4107
	$\mathbf{p_2}$	-0.0099	0.1043	0.0032	0.1481	-0.0199	-0.2570	-0.7756	-0.5468
3	$\mathbf{v_1}(\mathbf{p_1})$	0.0137	-0.0534	0.1166	-0.1019	-0.0029	-0.8773	0.1861	0.4107
	$\mathbf{v_2}$	0.0134	0.0341	0.1318	-0.0131	-0.0038	-0.3770	-0.7725	-0.4924
	$\mathbf{v_3}$	-0.1085	0.3609	-0.4663	0.7815	-0.0102	-0.1724	-0.0017	0.0092
	$\mathbf{p_2}$	0.0035	-0.0557	0.0292	-0.0690	-0.0008	-0.5838	0.5399	0.5991

Example 1: Exact Null Space of One Dimension.
Figure 2a shows $n = 50$ poses generated by a RR dyad. In this example, as in

Table 2: Extracted mechanism dimensions for Examples 1–3

Ex.	**Component**	x_1	x_2	a_1	a_2	r
1, 2, 3	RR dyad 1	3.8919	-8.4984	7.4279	0.2100	4.7786
2	RR dyad 2	10.5152	0.3210	14.9360	-2.0085	10.8282
3	triad	15.7490	-8.2684	19.5157	0.2373	[1.9931, 6.5740]

the two following ones, we assume that we do not have any prior knowledge of the existence of the underlying constraints and employ the null-space analysis method outlined in Sec. 4 to acquire the circular constraint from the pose data. It yields only one zero singular value ($\lambda_1 = 0$), and the corresponding singular vector, $\mathbf{v}_1$, has been verified to satisfy both quadratic constraints (4), defining the dyad vector $\mathbf{p} = \mathbf{v}_1$ (Table 1). Thus, there exists a *RR* dyad with parameters listed in Table 2 and illustrated in Fig. 2a.

Example 2: Exact Null Space of Two Dimensions.
For $n = 25$ poses generated by a planar *RRRR* linkage sharing the left dyad of Example 1 (Fig. 2b), the null-space analysis yields two zero singular values ($\lambda_1 = \lambda_2 = 0$) with singular vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ (Table 1). However, $\mathbf{v}_1$ differs from that in Example 1, and neither $\mathbf{v}_1$ nor $\mathbf{v}_2$ satisfies the quadratic constraints in Eq. (4). In this case, we expand to a 3D approximate null space by including singular vectors $\mathbf{v}_3$ for $\lambda_3 = 0.1604$. A general vector in this space, $\mathbf{p} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3$ (where $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$), is substituted into Eq. (4), yielding four real solutions. Filtering for zero algebraic residual error $\|[A]\mathbf{p}\|$ isolates the physical constraints, giving $(\alpha_1, \alpha_2, \alpha_3)$ sets of $(0.4082, 0.9129, 0)$ and $(0.8337, -0.5522, 0)$. This leads to dyads $\mathbf{p}_1$ (matching Example 1) and $\mathbf{p}_2$, whose dimensions are listed in Table 2. Together, they form the 1-DOF *RRRR* linkage in Fig. 2b.

Example 3: Approximate Null-Space of Three Dimensions.
A 45-pose dataset (Fig. 3) combines the 25 poses of the planar *RRRR* linkage in Example 2 with 20 new poses where the right dyad length varies by $\pm 10\%$. The null-space analysis yields only one zero singular value ($\lambda_1 = 0$), producing a $\mathbf{v}_1$ identical to Example 1 (Table 1). Expanding to a 3D approximate null space with singular vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ ($\lambda_2 = 0.1093, \lambda_3 = 1.1291$), and substituting the analogous linear combination for $\mathbf{p}$ into Eq. (4) yields four real solutions. We select the two solutions that minimize the algebraic residual error $\|[A]\mathbf{p}\|$, yielding $(\alpha_1, \alpha_2, \alpha_3)$ sets of $(1, 0, 0)$ and $(0.8722, -0.4889, 0.0172)$. These define $\mathbf{p}_1$ and $\mathbf{p}_2$, where $\mathbf{p_1} = \mathbf{v_1}$, as expected. The second solution, $\mathbf{p}_2$, defines a circular ring constraint representing a dyad with variable link length, which is equivalent to a *RPR* triad. The center of the ring, $\mathbf{a} = (a_1, a_2)$, and the moving pivot, $\mathbf{x} = (x_1, x_2)$, can be obtained from $\mathbf{p}_2$ as before; however, the radius is no longer a constant but varies in the range $[r_{min}, r_{max}]$. In this case, $[A]\mathbf{p}_2 \neq \mathbf{0}$, from which we can

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Fig. 3: A set of 45 poses constrained by a circle and a circular ring with a coupler link at the outer and inner circles of the ring.

identify the poses that correspond to the maximum and minimum values in the column vector $[A]\mathbf{p}_2$. These extreme values correspond to the poses at the extreme location relative to the triad. The two extreme locations of $\mathbf{x}$ in F can be obtained using $\mathbf{X} = [H]\mathbf{x}$. It follows that $r_{min} = |\mathbf{X}_{min} - \mathbf{a}|$ and $r_{max} = |\mathbf{X}_{max} - \mathbf{a}|$. The mechanisms in the two extreme configurations are shown in Fig. 3, with dimensions listed in Table 2.

6 Conclusions

This paper extends and refines our previous work for planar four-bar linkage synthesis in the framework of kinematic acquisition of geometric constraints. This null-space approach lends itself naturally to a unified framework for determining not only dimensions but also DOF of the underlying motion as well as the number of dyads or triads needed to fit the given pose data.

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