Complete Classification of Flat Increased Instantaneous Mobility-Singular Configurations of 4-R Planar Mechanism

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Abstract This paper proposes a methodology for classifying all the flat increased instantaneous mobility (IIM)-singular configurations of the 4-R (revolute) closed-loop planar mechanism. The constraint functions are expanded up to second-order via Taylor series and the conditions for existence of solutions are investigated. This facilitated establishing the connection between the lengths of links of the mechanism and its capability for second-order infinitesimal displacement through the flat IIM-singular configurations. The method can be extended to other mechanisms, such as a 5-R closed-loop planar mechanism.

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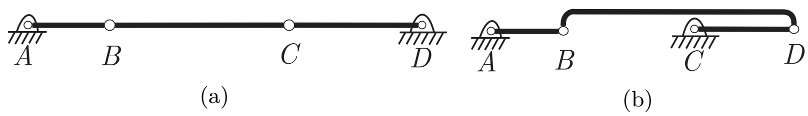


Fig. 1: Two flat configurations of 4-R mechanism in IIM-singularity: (a) cannot move out of the locked situation, (b) able to move out, i.e., "unlocked".

1 Introduction

Special configurations of closed-loop mechanisms have always attracted the attention of researchers, leading to enrichment of theoretical insights into the associated kinematic characteristics [1–3]. Singularities of mechanisms and classification of singularities into six types are reported in detail in [2]. One of the important types of singularity is the *increased instantaneous mobility* (IIM), *which occurs when the transitory or instantaneous mobility is higher than the full-cycle mobility of the kinematic chain.* There exists a class of special configurations of mechanisms called *flat configurations*. At a flat configuration of a planar 4-R mechanism, all the revolute pairs of the mechanism lie on a common line. Such a configuration is also termed as *uncertainty configuration* [4]. The flat configurations of 4-R mechanism suffer an IIM-singularity[1] in the sense that the mechanisms exhibit an instantaneous *gain* of one additional degree of freedom (d.o.f.). Two such flat IIM-singular configurations of a 4-R planar mechanism are shown in Fig. 1. Intuitively, the mechanism in Fig. 1(a) cannot move out of the flat IIM-singular configuration while the second one in Fig. 1(b) can. The existing literature geometrically [4] and analytically [3] concur that both the mechanisms have an instantaneous gain of d.o.f. In other words, these are indistinguishable up to the first-order of *instantaneous kinematics*. Therefore, it becomes necessary to study the second-order properties of the system. To distinguish the above two mechanism assemblies, one needs to first understand the concept of instantaneous kinematics and its order.

Broadly, the mobility or d.o.f. of a mechanism is defined as the topological dimension of its configuration space. The terms "global mobility", "local mobility" and "instantaneous mobility" are used in the context of mechanisms [5]. At any configuration of a mechanism, *local mobility* is the local dimension of its configuration space. *Global dimension* is the maximum of local dimensions over all the points of the configuration space.

Earlier researchers [5] defined the *infinitesimal mobility* of a configuration as the dimension of the tangent space to the corresponding point in the

[1] The detailed mathematical treatment for synthesis of all the IIM-singular configurations of planar 4-R mechanism is not in the scope of the present work. However, Section 3 of the present work contains a mathematical result showing that the flat configurations indeed exhibit an increased instantaneous mobility.

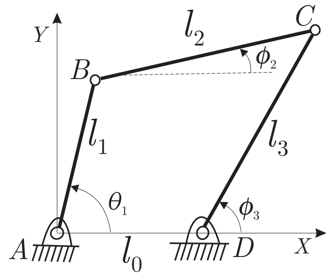


Fig. 2: Kinematic diagram of a 4-R planar mechanism.

configuration space; this, in effect, is the first-order instantaneous mobility. In the existing literature, reports pertaining to higher-order instantaneous mobility using constraint equations [6] and screws and their derivatives [7] are available. However, an in-depth analysis of any specific mechanism is not readily available. Moreover, the method introduced in [7] is more of a mobility characterization scheme and not suitable for complete structured analysis of all variants of a mechanism. A school of engineers studied structures and mechanical assemblies with infinitesimal mobility by means of analytical statics [8].

The present work aims to classify *all* the flat IIM-singular configurations of 4-R planar mechanism. A methodology is proposed that allows an analytical and comprehensive characterization of all the flat IIM-singular configurations, in which, these are classified into two groups, namely, (a) ones that can pass through these singularities by virtue of the instantaneous mobility mentioned above, and (b) those which cannot pass and get locked in the assembled configuration. The corresponding relations among the link lengths for realizing the configurations in each group are also obtained.

The rest of the paper is organized as follows. Section 2 of the paper proposes a general mathematical formulation for ascertaining the second-order displacement characteristics of a mechanism in any configuration. The detailed analysis of the flat IIM-singular configurations of the 4-R mechanism is presented in Section 3. The contributions of this paper, along with potential directions for future extensions, are presented in Section 4.

2 Mathematical Development

Let $\eta_1(\mathbf{q}) = 0$ and $\eta_2(\mathbf{q}) = 0$ be the loop-closure equations corresponding to the X and Y components, where $\mathbf{q}$ represents the vector comprising the joint variables. For the 4-R mechanism shown in Fig. 2, $\mathbf{q} = [\theta_1, \phi_2, \phi_3]^\top$. The mechanism is initially assembled at a configuration designated by $\mathbf{q}_0$, implying $\eta_i(\mathbf{q}_0) = 0,\ i = 1, 2$. Expanding the constraint equations $\eta_1(\mathbf{q}) = 0$

and $\eta_2(\mathbf{q}) = 0$ up to the second-order term of Taylor series about $\mathbf{q}_0$ yields:

$$\eta_i(\mathbf{q}_0 + \delta\mathbf{q}) \approx \eta_i(\mathbf{q}_0) + \boldsymbol{J}_i\delta\mathbf{q} + \delta\mathbf{q}^\top \boldsymbol{H}_i\delta\mathbf{q} = 0, \quad i = 1, 2. \tag{1}$$

Here, $\boldsymbol{J}_i = \boldsymbol{J}_i(\mathbf{q}_0) = \left[\frac{\partial\eta_i}{\partial\mathbf{q}}\right]$, $i = 1, 2$, are the respective Jacobian matrices of the constraint expressions at the configuration $\mathbf{q}_0$. Similarly, $\boldsymbol{H}_i = \boldsymbol{H}_i(\mathbf{q}_0) = \left[\frac{\partial^2\eta_i}{\partial \mathrm{q}_j \partial \mathrm{q}_k}\right]$, $i, j, k = 1, 2$, are the corresponding Hessian matrices at $\mathbf{q}_0$.

The mechanism configurations under consideration have the following kinematic characteristics:

(a) Mechanism can posses non-zero velocity state, i.e., the mechanism can remain assembled even for first-order infinitesimal displacements in the vicinity of the assembled configuration $\mathbf{q}_0$.

(b) However, the mechanism may or may not allow second-order infinitesimal displacements in the same neighbourhood of $\mathbf{q}_0$.

Statement (a) above implies that the system of constraint equations retain a consistent solution even after a first-order infinitesimal displacement (without enforcing a disassembly of the mechanism through such a displacement). Mathematically, it means that the equations

$$\boldsymbol{J}_1\delta\mathbf{q} = 0, \tag{2}$$

$$\boldsymbol{J}_2\delta\mathbf{q} = 0, \tag{3}$$

share a consistent non-trivial solution.

As the mechanical assemblability continues through higher-orders of instantaneous kinematics, the corresponding terms in the Taylor expansion should vanish in Eq. (1).

Statement (b) implies that the mechanism should admit the infinitesimal displacement $\mathbf{q}$ up to the first-order; hence, Eqs. (2, 3) should be satisfied simultaneously. However, the feasibility of infinitesimal displacements of higher order is of interest. From Eqs. (1-3), it can be said that if the following equations

$$\delta\mathbf{q}^\top \boldsymbol{H}_1\delta\mathbf{q} = 0, \tag{4}$$

$$\delta\mathbf{q}^\top \boldsymbol{H}_2\delta\mathbf{q} = 0, \tag{5}$$

admit the same non-trivial solution $\delta\mathbf{q}$, then the mechanism (see, e.g., Fig. 1(b)) allows second-order infinitesimal displacement in the vicinity of $\mathbf{q}_0$. In such a case, the mechanism can pass through $\mathbf{q}_0$ with finite, non-zero acceleration. Conversely, if no such $\delta\mathbf{q}$ exists, the mechanism remains locked (like the one shown in Fig. 1(a)) at $\mathbf{q}_0$, inspite of being capable of producing non-zero velocities at $\mathbf{q}_0$, instantaneously. These conditions are analyzed below for a 4-R mechanism.

Table 1: Classification of 8 flat IIM-singular configurations of the 4-R mechanism.

	$\mathbf{q}_0 = [\theta_1, \phi_2, \phi_3]^\top$			Relation between link lengths	Definiteness of $\boldsymbol{M}$	Classification
	θ_1	ϕ_2	ϕ_3			
Case 1	π	0	π	$l_0 + l_1 - l_2 - l_3 = 0$	Indefinite	Unlocked
Case 2	0	π	π	$l_0 - l_1 + l_2 - l_3 = 0$	Indefinite	Unlocked
Case 3	0	0	0	$l_0 - l_1 - l_2 + l_3 = 0$	Indefinite	Unlocked
Case 4	π	π	π	$l_0 + l_1 + l_2 - l_3 = 0$	Positive Definite	Locked
Case 5	π	0	0	$l_0 + l_1 - l_2 + l_3 = 0$	Positive Definite	Locked
Case 6	0	π	0	$l_0 - l_1 + l_2 + l_3 = 0$	Positive Definite	Locked
Case 7	0	0	π	$l_0 - l_1 - l_2 - l_3 = 0$	Negative Definite	Locked
Case 8	π	π	0	$l_0 + l_1 + l_2 + l_3 = 0$	Positive Definite	Infeasible Mechanism

3 Flat IIM-Singular Configurations of 4-R Mechanism

The present section applies the theory developed in Section 2 to classify the flat IIM-singular configurations of 4-R mechanism. The constraint equations for the 4-R mechanism shown in Fig. 2 are:

$$\eta_1(\mathbf{q}) \triangleq l_1 \cos\theta_1 + l_2 \cos\phi_2 - l_0 - l_3 \cos\phi_3 = 0, \tag{6}$$
$$\eta_2(\mathbf{q}) \triangleq l_1 \sin\theta_1 + l_2 \sin\phi_2 - l_3 \sin\phi_3 = 0. \tag{7}$$

The Jacobian matrices of the above constraint functions w.r.t. $\mathbf{q}$ are:

$$\boldsymbol{J}_1 = \Big[-l_1 \sin\theta_1 \ , -l_2 \sin\phi_2 \ , l_3 \sin\phi_3\Big] , \tag{8}$$

$$\boldsymbol{J}_2 = \Big[l_1 \cos\theta_1 \ , l_2 \cos\phi_2 \ , -l_3 \cos\phi_3\Big] . \tag{9}$$

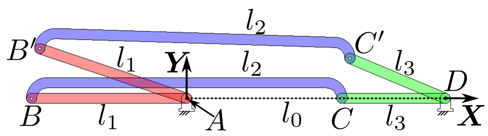
At a general (i.e., non-singular) configuration, the rank of the constraint Jacobian matrix $\boldsymbol{J} = [\boldsymbol{J}_1^\top \boldsymbol{J}_2^\top]^\top$ is two, hence, only, one of $\{\delta\theta_1, \delta\phi_2, \delta\phi_3\}$ can be independently specified; i.e., the mechanism has a local mobility of one, as expected.

The corresponding Hessian matrices $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are obtained as:

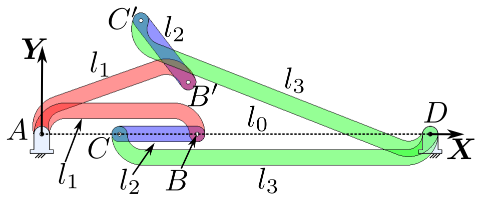
$$\boldsymbol{H}_1 = \mathrm{diag}(-l_1 \cos\theta_1, -l_2 \cos\phi_2, l_3 \cos\phi_3), \tag{10}$$

$$\boldsymbol{H}_2 = \mathrm{diag}(-l_1 \sin\theta_1, -l_2 \sin\phi_2, l_3 \sin\phi_3). \tag{11}$$

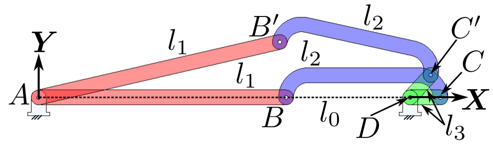
For the flat configurations, each of the angles $\{\theta_1, \phi_2, \phi_3\}$ is equal to either 0 or π. *Hence, both $\boldsymbol{J}_1$ and $\boldsymbol{H}_2$ are null matrices.* Since $\boldsymbol{J}_1$ is a null matrix, Eq. (2) degenerates to an identity and imposes no constraints on $\delta\mathbf{q}$, which has only to satisfy Eq. (3). This implies that the mechanism instantaneously



(a) Case 1: $l_0 + l_1 - l_2 - l_3 = 0$



(b) Case 2: $l_0 - l_1 + l_2 - l_3 = 0$



(c) Case 3: $l_0 - l_1 - l_2 + l_3 = 0$

Fig. 3: Flat IIM-singular configurations of the 4-R mechanism along with the corresponding relations among the link lengths. These configurations are in an unlocked state corresponding to Cases 1-3. The joints B and C move to B', and C', respectively, under motion from the flat IIM-singular configuration.

gains an additional d.o.f. at a flat configuration[2]; hence, flat configurations inherently exhibit an IIM type singularity. Now, Eq. (3) in the present scenario gives:

$$l_1 \cos\theta_1 \delta\theta_1 + l_2 \cos\phi_2 \delta\phi_2 - l_3 \cos\phi_3 \delta\phi_3 = 0; \tag{12}$$

$$\Rightarrow \delta\phi_3 = \frac{l_1 \cos\theta_1 \delta\theta_1 + l_2 \cos\phi_2 \delta\phi_2}{l_3 \cos\phi_3}. \tag{13}$$

Note that, the above equation could have been solved for $\delta\phi_2$ instead, where $\delta\phi_2 = (l_3 \cos\phi_3 \delta\phi_3 - l_! \cos\theta_1 \delta\theta_1)/(l_2 \cos\phi_2)$, and so on. Since the terms in Eq. (1) vanish up to first order, only the second-order terms remain in the Taylor series expansion of the constraint functions. If there exists a non-trivial $\delta\mathbf{q}$ which satisfies Eq. (4), then the mechanism can have second-order infinitesimal displacement (Eq. (5) has now degenerated into an identity as $\boldsymbol{H}_2$ became null). If Eq. (4) is not satisfied by any non-trivial $\delta\mathbf{q}$, then second-order infinitesimal displacement is not possible in the vicinity of $\mathbf{q}_0$. The

[2] One can start with the condition of rank deficiency of $\boldsymbol{J} = [\boldsymbol{J}_1^\top \boldsymbol{J}_2^\top]^\top$. In conjunction with the constraint equations it can be deduced that each of the angles $\{\theta_1, \phi_2, \phi_3\}$ being equal to either 0 or π are valid solutions to this condition. These details are skipped here for brevity.

quadratic form in Eq. (4) can be written, using the Hessian matrix in Eq. (10), as:

$$\delta\mathbf{q}^\top \boldsymbol{H}_1 \delta\mathbf{q} = \begin{bmatrix}\delta\theta_1 & \delta\phi_2 & \delta\phi_3\end{bmatrix} \begin{bmatrix} -l_1\cos\theta_1 & 0 & 0 \\ 0 & -l_2\cos\phi_2 & 0 \\ 0 & 0 & l_3\cos\phi_3 \end{bmatrix} \begin{bmatrix}\delta\theta_1 \\ \delta\phi_2 \\ \delta\phi_3\end{bmatrix}, \quad (14)$$

$$\Rightarrow \delta\mathbf{q}^\top \boldsymbol{H}_1 \delta\mathbf{q} = -l_1\cos\theta_1(\delta\theta_1)^2 - l_2\cos\phi_2(\delta\phi_2)^2 + l_3\cos\phi_3(\delta\phi_3)^2. \quad (15)$$

Now, $\{\delta\theta_1, \delta\phi_2, \delta\phi_3\}$ in the above equations are not all independent. Using Eq. (13), $\delta\phi_3$ can be written in terms of the two independent variables: $\delta\theta_1$, and $\delta\phi_2$. Thus, ensuring first-order consistency with the instantaneous kinematic constraints, Eq. (15) reduces to:

$$\delta\mathbf{q}^\top \boldsymbol{H}_1 \delta\mathbf{q} = a(\delta\theta_1)^2 + 2b\delta\theta_1\delta\phi_2 + c(\delta\phi_2)^2, \quad \text{where} \quad (16)$$

$$a = \frac{l_1\cos\theta_1}{l_3\cos\phi_3}(l_1\cos\theta_1 - l_3\cos\phi_3), \quad b = \frac{l_1\cos\theta_1}{l_3\cos\phi_3} l_2\cos\phi_2, \quad \text{and} \quad (17)$$

$$c = \frac{l_2\cos\phi_2}{l_3\cos\phi_3}(l_2\cos\phi_2 - l_3\cos\phi_3). \quad (18)$$

The expression in Eq. (16) can be reduced to a quadratic form of two *independent variables* $\{\delta\theta_1, \delta\phi_2\}$ as: $\delta\mathbf{q}^\top \boldsymbol{H}_1 \delta\mathbf{q} = \begin{bmatrix}\delta\theta_1 & \delta\phi_2\end{bmatrix} \boldsymbol{M} \begin{bmatrix}\delta\theta_1 \\ \delta\phi_2\end{bmatrix}$, where $\boldsymbol{M} = \begin{bmatrix} a & b \\ b & c\end{bmatrix}$.

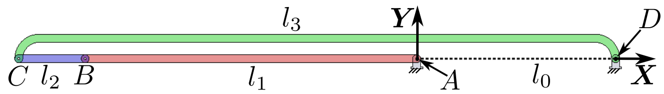
The leading principal minors of matrix $\boldsymbol{M}$ are obtained as:

$$d_1 = \frac{l_1\cos\theta_1}{l_3\cos\phi_3}(l_1\cos\theta_1 - l_3\cos\phi_3), \quad (19)$$

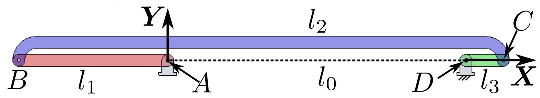
$$d_2 = -\frac{l_1 l_2\cos\theta_1\cos\phi_2}{l_3\cos\phi_3}(l_1\cos\theta_1 + l_2\cos\phi_2 - l_3\cos\phi_3). \quad (20)$$

The capability (or inability) of the 4-R mechanism at a flat IIM-singular configuration to generate second-order infinitesimal displacements depends upon the definiteness of the matrix $\boldsymbol{M}$. There are a total of 8 possible flat configurations, as each of the joint angles $\{\theta_1, \phi_2, \phi_3\}$ can be either 0 or π. The definiteness of $\boldsymbol{M}$ for each of these cases (see Table 1) is ascertained by determining the algebraic signs of the leading principal minors given in Eqs. (19, 20) and applying Sylvester's criteria [9] for definiteness. The configurations of the 4-R mechanisms are shown in Figs. 3, 4.

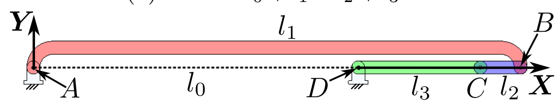
For example, consider the case when the joint angles are $\mathbf{q}_0 = [0, \pi, \pi]^\top$. In this case, $d_1 = -\frac{l_1(l_1+l_3)}{l_3} < 0$ (since all the link lengths are positive) and $d_2 = -\frac{l_1 l_2(l_1-l_2+l_3)}{l_3}$. Now, $\mathbf{q}_0 = [0, \pi, \pi]^\top$ satisfies the constraint Eq. (6) yielding



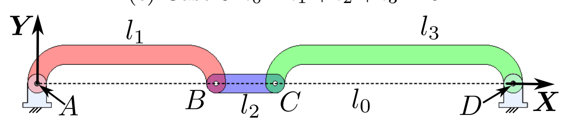
(a) Case 4: $l_0 + l_1 + l_2 - l_3 = 0$



(b) Case 5: $l_0 + l_1 - l_2 + l_3 = 0$



(c) Case 6: $l_0 - l_1 + l_2 + l_3 = 0$



(d) Case 7: $l_0 - l_1 - l_2 - l_3 = 0$

Fig. 4: Flat IIM-singular configurations of the 4-R mechanism along with the corresponding relations among the link lengths. These configurations are in a locked state corresponding to Cases 4-7.

$l_0 - l_1 + l_2 - l_3 = 0$. Therefore, $l_1 - l_2 + l_3 = l_0 > 0$, which implies $d_2 < 0$. Thus, both the leading principal minors are strictly negative, which characterizes $\boldsymbol{M}$ as an indefinite matrix. Hence, for this case, $\delta\mathbf{q}^\top \boldsymbol{H}_1 \delta\mathbf{q} = 0$ admits a non-trivial solution. The kinematic implication of this is that the mechanism can have second-order infinitesimal displacements in the neighbourhood of $\mathbf{q}_0$ and consequently, can move out of the flat IIM-singular configuration. It is therefore and is labeled as *unlocked* (see Case 2) in Table 1.

All the cases are analyzed similarly as discussed above. The definiteness of the matrix $\boldsymbol{M}$ and the corresponding characteristic of motion capability are listed in Table 1. The mechanisms corresponding to all the cases have been built in a CAD software[3] to check for the movability from the flat configuration. The locked or unlocked behaviour associated with each case, as reported in Table 1, is seen to be true. These 4-R mechanisms are shown in Figs. 3, and 4.

[3] The assembly files built in SOLIDWORKS 2023 [10] have been shared via GitHub: https://github.com/Bibek67/Four_bar_locking_unlocking_configurations_CAD_model.git.

4 Conclusion

The present work investigated the feasibility of second-order infinitesimal motions of all 8 flat 4-R mechanisms which are in an IIM-singularity. It is shown that 4 flat configurations of 4-R mechanism do not allow it to move out of such a gain singular configuration. Appropriate relations among the link lengths of the mechanism, which dictate the locked or unlocked behaviour, are also derived for all the mechanisms. The traditional classification of 4-R mechanisms based on their rotational capability should also include the locked mechanisms apart from the usual Grashof, non-Grashof, and change point mechanisms. The locked configurations with their small ranges of motion may be of practical engineering utility in the design framework of flexure mechanisms. Further work is under progress in extending the present method to the singular configurations of 5-R closed-loop and other planar mechanisms.

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