On the Singularities of Robotic Stiffness Matrices

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Abstract In this paper, we analyze and explain the general Cartesian robotic stiffness matrix conditions that generate singularities in the system for a robot performing stiffness or impedance control related tasks, which not only come from the kinematics itself. From first principles, we show that the natural frequencies of a robotic manipulator can be used as a tool when handling or dealing with singularities. Simulation results validate and illustrate the theory and bounding conditions encountered.

1 Introduction

Robotic stiffness and impedance control have been widely used to control the dynamics of interaction of a robot and the environment. In stiffness control, a multidimensional stiffness matrix is imposed in order to maintain the forces and torques to a certain level[2, 12].

This multidimensional relation grows out of the classical scalar mass-spring system constrained to a motion in one direction, with the particular control criteria that define and characterize this type of systems subject to external excitation or free vibration with initial conditions [8].

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In terms of applicability, when implementing this kind of mass-damper-spring system, in order to maintain stability and successfully achieve the intended tasks, an appropriate amount of damping must also be imposed. Additionally, as the robot moves inside its workspace, one must account for singular configurations in which the Jacobian matrix loses rank and affect the motion of mechanisms, and have been greatly studied by many authors [13].

By the analysis presented in this paper, we propose to observe the natural frequencies of the multidimensional system in order to consequently handle singularities. We can see how the most critical (dominant) modes tend to zero as the robot approaches a singularity. We also present certain conditions different from the kinematic singularities, that give rise to stiffness singularities, where the natural frequency being tracked vanishes.

2 Modeling Stiffness

2.1 Background

Many robotic tasks such as assembly, machining, wiping, among others, rely on an interaction model that relates an end-effector wrench $\mathbf{w}$ to a small pose displacement $\delta\mathbf{x}$ through an imposed task space stiffness matrix $\mathbf{K}_d$ [10]:

$$\mathbf{w} = \mathbf{K}_d\, \delta\mathbf{x}.$$

In a conservative mechanical system, $\mathbf{K}_d$ is typically symmetric because it is the Hessian of a scalar potential energy $U(\delta\mathbf{x})$,

$$\mathbf{w} = \nabla_{\delta\mathbf{x}} U(\delta\mathbf{x}), \qquad \mathbf{K}_d = \frac{\partial \mathbf{w}}{\partial(\delta\mathbf{x})} = \nabla^2_{\delta\mathbf{x}} U(\delta\mathbf{x}), \qquad \mathbf{K}_d = \mathbf{K}_d^\top.$$

This symmetry is also commonly enforced at the control level in Cartesian stiffness/impedance control [1, 4], where the commanded wrench is chosen as

$$\mathbf{w}_c = \mathbf{K}_d\,(\mathbf{x}_d - \mathbf{x}) + \mathbf{C}_C\,(\dot{\mathbf{x}}_d - \dot{\mathbf{x}}), \qquad \mathbf{K}_d = \mathbf{K}_d^\top \succ 0,$$

so that the closed-loop interaction behaves like a passive mass–spring–damper around the desired pose $\mathbf{x}_d$.

However, practical systems may exhibit effective stiffness matrices that are not symmetric ($\mathbf{K}_d \neq \mathbf{K}_d^\top$) due to nonconservative effects (e.g., follower forces, frictional contact, preload), active control coupling, and model reductions. Understanding when such non-symmetry is negligible and when it induces energy injection is essential for safe, high-performance physical interaction.

2.2 From potential energy to symmetric stiffness

A common control objective is to realize a *desired symmetric Cartesian stiffness* $\mathbf{K}$ when performing robotic stiffness/impedance related tasks.

Because $\mathbf{K}$ is symmetric positive definite, the corresponding elastic energy $V(\mathbf{x}) = \frac{1}{2}(\mathbf{x}-\mathbf{x}_d)^\top \mathbf{K}(\mathbf{x}-\mathbf{x}_d)$ is well-defined and helps ensure passivity when combined with symmetric damping. This energy-based interpretation is one of the main reasons symmetry is often enforced in stiffness control.

Recent work also explores data-driven selection of stiffness parameters (including non-diagonal structures) for contact-rich tasks [9].

However, in practice, symmetric stiffness may not always be realized, or a non-symmetric matrix may be actively imposed from control due to intended interaction behaviors that may arise, at the expense of losing the conservative property, and the need of imposing sufficient damping to ensure stability.

2.3 Joint stiffness and natural frequencies

When a desired stiffness $\mathbf{K}$ is specified in Cartesian space, an equivalent joint-space stiffness $\mathbf{K}_q$ can be derived via the Jacobian $\mathbf{J}(\mathbf{q})$ under small deflections[1]. Using $\delta\mathbf{x} = \mathbf{J}\,\delta\mathbf{q}$ and $\boldsymbol{\tau} = \mathbf{J}^\top \mathbf{w}$, the conservative (symmetric) mapping yields

$$\boldsymbol{\tau} = \mathbf{K}_q\,\delta\mathbf{q}\,; \qquad \mathbf{K}_q = \mathbf{J}^\top \mathbf{K}\,\mathbf{J}. \tag{1}$$

which can be combined with the joint-space inertia $\mathbf{M}(\mathbf{q})$ to characterize local natural frequencies [6]. In particular, if λ_i are the eigenvalues of the generalized eigenproblem, and ϕ_i are the eigenvectors, such that

$$\mathbf{K}_q\,\boldsymbol{\phi}_i = \lambda_i\,\mathbf{M}\,\boldsymbol{\phi}_i, \tag{2}$$

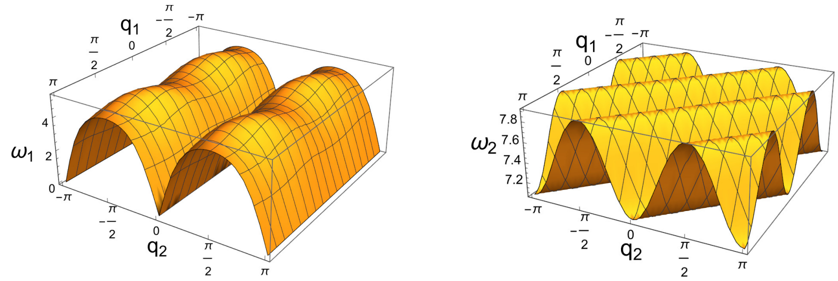
then, the (undamped) natural frequencies are

$$\omega_i = \sqrt{\lambda_i}.$$

As it can be seen from Equations (1) and (2), the natural frequencies of the joint system depend on the eigenvalues of the product $\mathbf{M}^{-1}\,\mathbf{K}_q$, more explicitly, involving the (chosen) elements of the original Cartesian stiffness matrix and the configuration dependent Jacobian matrix

$$\mathbf{D} = \mathbf{M}^{-1}\left(\mathbf{J}^\top \mathbf{K}\,\mathbf{J}\right) \tag{3}$$

[1] The mapping $\mathbf{K}_q = \mathbf{J}^\top \mathbf{K}\,\mathbf{J}$ is a (conservative) congruence transformation that preserves symmetry/exactness of stiffness matrices under coordinate changes only under certain conditions; see [1] for a detailed discussion.



(a) $w_1(q_1, q_2)$ (b) $w_2(q_1, q_2)$

Fig. 1: Plot samples of natural frequencies for a 2R robot and given parameters as functions of q_1 and q_2. Diagonal $\mathbf{K}$ case (all in SI units).

In the following, we will analyze the natural frequencies of a serial robot in the joint space in certain different cases, where the behavior of the system depends on the different sources of rank deficiencies of $\mathbf{K}_q$, as it will be explained.

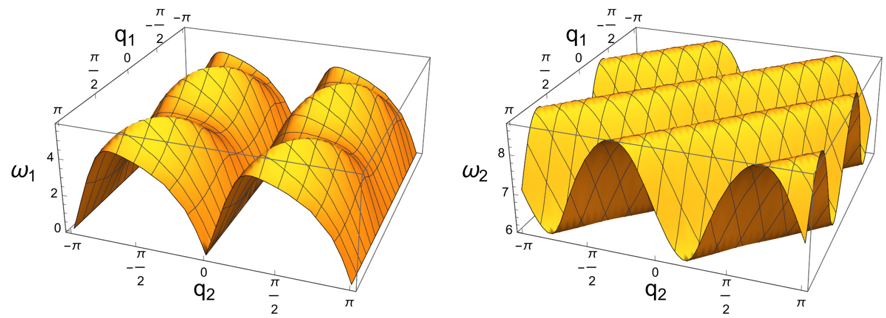
2.4 Diagonal Cartesian stiffness

In this case, we obtain a symmetric, positive (semi)definite joint stiffness matrix $\mathbf{K}_q = \mathbf{K}_q^\top$ with real eigenvalues, including 0 (rank deficiency and constrained motion in certain directions) near kinematic singularities, and in redundant manipulators with n joints and an m-dimensional task space ($n > m$, $\mathbf{J}(\mathbf{q}) \in \mathbb{R}^{m \times n}$) [10, 3], as expected from the structure of Equation (1). The eigenvalues of $\mathbf{D}$ will be real, as the mass matrix $\mathbf{M}(\mathbf{q})$ is positive definite, and the rank deficiency will only come from the Jacobian matrix.

For example, for a 2R planar robot with point masses m_1 and m_2 at the tip of each link l_1 and l_2, and generalized coordinates q_1 and q_2, respectively, we can see in Figure 1 how a positive definite diagonal Cartesian $\mathbf{K}$ ($k_{11} = 50$, $k_{22} = 40$) generates a system in the joint space that has two natural frequencies, one of them (ω_1) approaching zero as the robot approaches a kinematic singularity.

2.5 Symmetric Cartesian stiffness

Here we analyze the joint-space system obtained from a (coupled) symmetric Cartesian stiffness matrix $\mathbf{K} = \mathbf{K}^\top$. In general, a symmetric stiffness matrix can be written as



(a) $w_1(q_1, q_2)$ (b) $w_2(q_1, q_2)$

Fig. 2: Plot samples of natural frequencies for a 2R robot and given parameters as functions of q_1 and q_2. Symmetric (coupled) $\mathbf{K}$ case (all in SI units).

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1m} \\ k_{12} & k_{22} & k_{23} & \cdots & k_{2m} \\ k_{13} & k_{23} & k_{33} & \cdots & k_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{1m} & k_{2m} & k_{3m} & \cdots & k_{mm} \end{bmatrix},$$

where the diagonal terms $k_{11}, k_{22}, \ldots, k_{mm}$ represent the stiffness components in each Cartesian direction, and the off-diagonal terms k_{ij} (e.g., $k_{12}, k_{13}, k_{23}, \ldots, k_{1m}$) capture coupling. In this case, $\mathbf{K}_q = \mathbf{J}^\top \mathbf{K}\,\mathbf{J}$ remains symmetric; however, its rank and eigenvalues still depend on $\mathbf{J}(\mathbf{q})$ (and thus on singularities and redundancy), and on the specific chosen elements of $\mathbf{K}$ as we shall see later below.

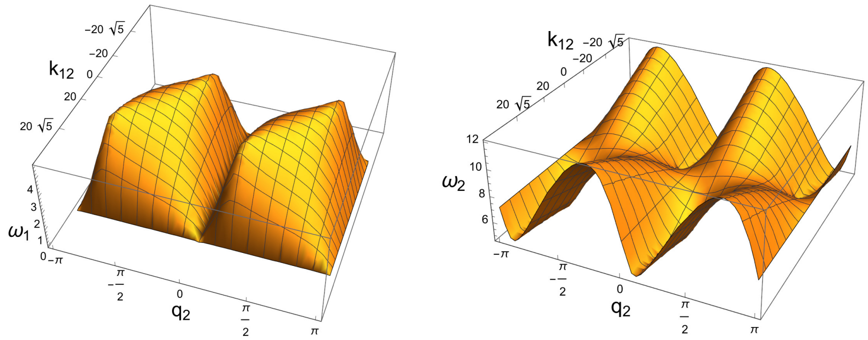
For example, as in the previous subsection, in the following, we will compute and see how the natural frequencies of the 2R robot system behave, this time using a positive definite symmetric Cartesian $\mathbf{K}$ with off-diagonal elements.

In Figure 2, it is shown that ω_1 and ω_2 are continuous for all values of q_1 and q_2. As the robot approaches a kinematic singularity, one of the natural frequencies (ω_1) tends to 0.

The positive definiteness property of the Cartesian stiffness matrix depends on the relative values between its diagonal and off-diagonal elements. A convenient sufficient-and-necessary condition for symmetric matrices is the positivity of all leading principal minors [11]:

$$\det\big(\mathbf{K}_{1:k,1:k}\big) > 0, \qquad k = 1, 2, \ldots, m, \tag{4}$$

where $\mathbf{K}_{1:k,1:k}$ denotes the $k \times k$ leading principal submatrix of $\mathbf{K}$.
Once again, let's take a look at the simple case of the 2R robot, in which the XY Cartesian stiffness matrix has the following form



(a) $w_1(k_{12}, q_2)$ (b) $w_2(k_{12}, q_2)$

Fig. 3: Plot samples of natural frequencies for a 2R robot and given parameters as functions of k_{12} and q_2. Symmetric (coupled) $\mathbf{K}$ case (all in SI units).

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix},$$

and the positivity of all leading principal minors condition establishes

$$|k_{12}| < \sqrt{k_{11}k_{22}}$$

which tells us that the task coupling cannot be 'stiffer' than the square root of the product of the axes being coupled. For example, considering $k_{11} = 50$, and $k_{22} = 40$, we know that the coupling element $|k_{12}| < 20\sqrt{5}$.

As it is illustrated in Figure 3, where both natural frequencies of the planar 2R system are plotted as functions of both sources of possible rank loss of $\mathbf{K}_q$: element k_{12} of $\mathbf{K}$ and joint q_2. As the robot approaches a kinematic singularity, one of the natural frequencies tends to 0. At the same time, it can be seen that as $\mathbf{K}$ no longer holds the positivity of all leading principal minors condition ($|k_{12}| >= 20\sqrt{5}$), the natural frequency vanishes since the corresponding eigenvalue of $\mathbf{D}$ in Equation (3) is no longer real.

2.6 Special case of singularity

The natural frequencies of a multidimensional robot manipulator in the joint space are a function of the configuration, and are given by the eigenvalues of the matrix resulting from the product shown in Equation (3), so since it is always assumed that the mass matrix is positive definite by the first structural law of the equation of motion [10], once the positive definiteness of the joint stiffness matrix $\mathbf{K}_q$ is ensured, we will have a positive definite system. However, in special cases, like when considering a negligible mass in one of the links, we may encounter additional conditions. For example, if $m_1 \approx 0$ for the first link (l_1) of a 2R robot with point masses at the end of each link (a common practical case) the expression for the mass matrix

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix} + \mathbf{J}^\top m_2 \, \mathbf{J}$$

tells us that it becomes rank deficient at the same conditions of the kinematic singularities of the robot. In this case, due to the form of the transformations and the product involved, the natural frequencies are no longer a function of $\mathbf{q}$, they become constant throughout the workspace (except at kinematic singularities), which is not very intuitive.

$$\mathbf{D} = \frac{1}{m_2} \mathbf{K}$$

3 Discussion

Using theory of mechanical vibrations, we are able to obtain the natural frequencies of robotic manipulators performing stiffness/impedance control related tasks. We have seen how ω's change or vanish as the robot configuration, parameters and its dynamic terms vary. We infer that this tool can be used to track, avoid and/or induce singularities for different robotic systems regardless of structure, redundancy conditions, or other characteristics.

The off-diagonal stiffness elements bounds shown in Figure 3 are analytically related to the stiffness ellipsoid being imposed in the Cartesian space by the elements in $\mathbf{K}$. The diagonal elements set the size of the ellipsoid along the coordinate axes, while the symmetric off-diagonal elements rotate and skew the ellipsoid. Positive definiteness requires that the principal minors be non-negative, so that every 2D slice of the ellipsoid must itself be an ellipse. If any pair violates the bound, one eigenvalue becomes negative, and the ellipsoid breaks.

On the other hand, the stiffness bounds encountered can be interpreted as analogous or can be explained using Hooke's law in continuum mechanics [5]

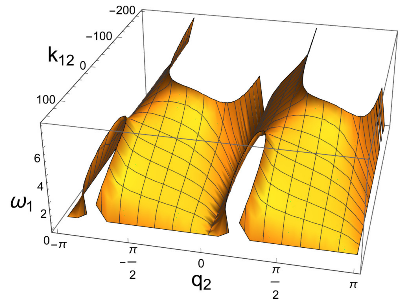
$$\boldsymbol{\sigma} = \mathbf{C} \, \boldsymbol{\varepsilon}$$

where the constitutive matrix $\mathbf{C}$ is symmetric positive definite for stable materials. As a consequence, its 2×2 leading principal minor implies

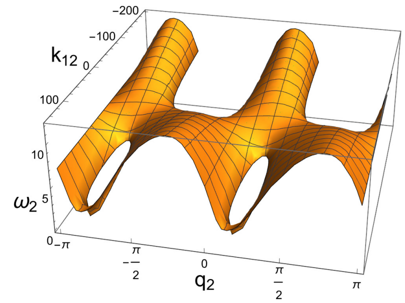
$$|C_{12}| < \sqrt{C_{11} C_{22}}$$

similar to the condition for the robotic Cartesian stiffness $\mathbf{K}$. Additionally, Poisson ratio can also be described as a coupling term, which is also bounded.

There might also be the need or task constraint to impose or deal with a non-symmetric stiffness matrix. This case no longer leads to a conservative system. A non-symmetric Cartesian (or joint) $\mathbf{K}$ can be decomposed into a symmetric and a skew-symmetric part



(a) $w_1(k_{12}, q_2)$



(b) $w_2(k_{12}, q_2)$

Fig. 4: Plot samples of natural frequencies for a 2R robot and given parameters as functions of k_{12} and q_2. Non-symmetric (coupled) $\mathbf{K}$ case (all in SI units).

$$\mathbf{K} = \mathbf{K}_s + \mathbf{K}_a \ ; \ \mathbf{K}_s = \frac{1}{2}(\mathbf{K} + \mathbf{K}^\top) \ ; \quad \mathbf{K}_a = \frac{1}{2}(\mathbf{K} - \mathbf{K}^\top)$$

where $\mathbf{K}_s$ is the symmetric part, and $\mathbf{K}_a$ is the skew-symmetric part of $\mathbf{K}$. This $\mathbf{K}_s$ matrix must be positive (semi)definite, or satisfy the positivity of all leading principal minors condition, while the skew-symmetric $\mathbf{K}_a$ matrix corresponds to the non-conservative term that is not passive and must be dealt with in a different manner, for example with an appropriate set of damping that ensures stability and desired dynamic behavior during the robotic task [7]. In addititon, given the structure of $\mathbf{K}_a$, this generates further considerations and conditions for rank deficiency.

As it can be seen in Figure 4, for the case of a planar 2R robot considering a non-symmetric stiffness matrix, its natural frequencies as functions of the non-diagonal element k_{12} (while keeping element k_{21} at a constant value different from zero) and joint q_2 have a different behavior than the one shown for the symmetric case, but including bounding conditions as well.

4 Conclusion and Future Work

In this paper, we have analyzed and shown results for different sources and conditions of singularities of stiffness matrices in robotic stiffness and impedance control in the Cartesian space.

We have established a tool that allows us to track the natural frequencies of a robotic system and will assist in the handling of singularities.

By having a better understanding of the singularity conditions in a conservative dynamic system, we can lay the path to deal with and modulate the dynamic response of a non-conservative system analytically, and affecting specific modes of motion.

Acknowledgements This work was partially supported by the Beatriu de Pinós Postdoctoral Research grant 2023 BP 00183 from the Generalitat de Catalunya.

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