Analytical Determination of the Proximity of Two Coplanar Non-Intersecting Conic Sections with Applications in Robotics

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Abstract This paper presents a new analytical method for computing the proximity of two generic conic sections in the same plane. The problem is solved by reducing it to the determination of the roots of a 12-degree univariate polynomial, whose coefficients are obtained symbolically in closed form.

This problem has relevance to many robotic applications, such as tracking the distance between the links of a (planar) robot and objects in its environment. The proposed formulation is demonstrated via an application to the problem of whole arm manipulation by a 3-R robot.

1 Introduction

Determination of the shortest distance between (i.e., the *proximity of*) two rigid objects in the plane has many applications in science, engineering, and computer graphics. For instance, in the domain of robotic manipulations, it is helpful in detecting potential collisions between the links of a manipulator and objects in its environment [10] or collisions between the links of two manipulators [7]. In mobile robotics, such geometric computations help in avoiding collisions of a robot with obstacles in its surroundings [12], or with other robots [6] operating in a shared space. Further applications include whole-arm manipulations [13], and CAD [2].

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One of the generic approaches to the computation of proximity (in planar robotics) is to encapsulate the links of the robot (or the robot itself, in case of a mobile robot) by a convex geometric shape, such as a circle [8, 5, 1]. This abstraction simplifies subsequent computations very significantly, but only at the cost of rendering the results highly conservative, since not many objects in a real-life scenario exhibit circular symmetry.

In this context, elliptical shapes offer a reasonable improvement as a smooth and convex envelope of objects of interest, as they afford additional provisions for incorporating directional properties as well as variations in shapes of the encapsulated objects via their orientation and dimension parameters [11, 3]. As expected, these improvements come at the cost of greater computational complexity, which may explain the paucity of mentions of ellipses in such contexts in the literature. To the best of the authors' knowledge, only recently can one find some applications of elliptical envelopes in the domains of serial [7], [14] and parallel manipulators [9]. Intersection of two coplanar ellipses has been studied analytically in [7], while the analytical detection of their tangency has been addressed in [9]. The shortest distance between such a pair of ellipses has also been computed, but only via numerical methods; see, e.g., [14]. As far as analytical computation of the proximity of such ellipses is concerned, the authors' search of the existing literature discovered only one document [4], wherein the problem has been formulated analytically, establishing an important result: that this problem may be reduced to that of solving a 12-degree univariate polynomial equation. Interestingly, however, the authors were unable to find any application of this concept/result in the domain of robotics.

This paper presents a new analytical approach to the problem of determining the proximity of two coplanar ellipses[1]. Additionally, it does so in the *most generic setting possible*, namely, that of determining the proximity of two coplanar conics of *arbitrary nature, described implicitly via symbolic coefficients*. In other words, the same formulation may be applied to any combination of ellipses, hyperbolas, parabolas etc., without any modifications. It is established that this generalised problem can admit *at most 12 isolated* solutions in the complex plane, as mentioned in [15]. Proceeding further, the above formulation is applied to a problem of robotics, namely, whole arm manipulation of an elliptical object by a planar 3-R robot, whose links are modelled as ellipses. The results are verified numerically using a commercial software.

It may be noted here that the approach presented in this paper differs fundamentally from the one proposed in [4]. The formulation presented in [4] is specific to a pair of ellipses. Furthermore, the formulation relies on these ellipses being represented in their respective parametric forms. Therefore, it

[1] This paper restricts itself to the discussion of non-intersecting ellipses for the simple reason that there is no space for a comprehensive analysis of these cases in the present paper. Further, the cases of tangent or intersecting ellipses do not present any additional challenges, and may be dealt with greater ease than the generic case.

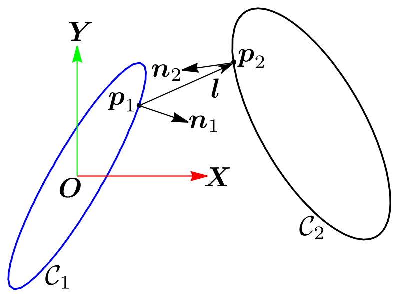


Fig. 1: Two conics $\mathcal{C}_1$ and $\mathcal{C}_2$ in a plane with their normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, respectively, and the vector $\boldsymbol{l}$ along the line joining $\boldsymbol{p}_1$, and $\boldsymbol{p}_2$

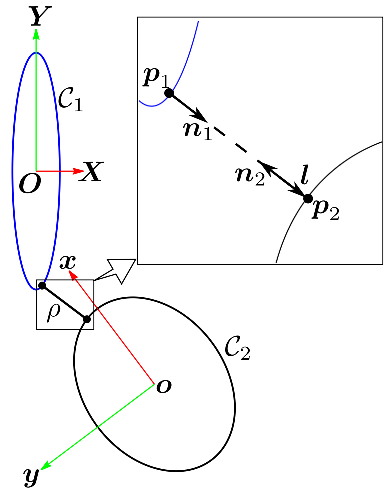


Fig. 2: Two non-intersecting conics $\mathcal{C}_1$ and $\mathcal{C}_2$ in the plane, with the normal vectors aligned along the line joining points $\boldsymbol{p}_1$, and $\boldsymbol{p}_2$

is imperative for this method to analyse a given pair of conics to ascertain their nature, as well as their geometric properties *explicitly*. Since such characterisations can only be performed when the coefficients of the conics in question are available in their *numerical* forms, or are related in a manner so as to render such preprocessing feasible, the said method cannot be used in situations wherein (a) the conics are not ellipses, or (b) the coefficients are arbitrary and non-numeric (i.e., symbolic), irrespective of whether the given conics are ellipses. These two aspects constitute the major advantages that the proposed method provides over the formulation presented in [4].

The rest of the paper is organised as follows. The mathematical formulation of the problem, in the setting of two generic (coplanar) conics, is presented in Section 2. These formulations are applied to one problem in robotics, as described above, in Section 3. The contributions of the paper and avenues for extensions of the present work are summarised in Section 4.

2 Mathematical formulation

The mathematical formulation for the computation of the proximity of two generic coplanar conics is presented in this section. The conics, illustrated as ellipses $\mathcal{C}_1$ and $\mathcal{C}_2$ in the XY-plane (see Fig. 1), are defined by the following equations:

$$f_1(x,y) := a_1x^2 + 2a_2xy + a_3y^2 + 2a_4x + 2a_5y + a_6 = 0, \tag{1}$$
$$f_2(x,y) := b_1x^2 + 2b_2xy + b_3y^2 + 2b_4x + 2b_5y + b_6 = 0, \tag{2}$$

where the coefficients $a_j, b_j,\ j = 1,\dots,6$, are arbitrary and symbolic. A pair of *proximal points*, i.e., $\boldsymbol{p}_i^* \in \mathcal{C}_i,\ i = 1,2$, is determined as an intermediate step to compute the proximity (denoted by ρ) of the conics (see Fig. 2). The derivation of the associated equations and the process of their solution are described below.

2.1 Derivation of the system of equations

A generic pair of points on $\mathcal{C}_1$ and $\mathcal{C}_2$ may be represented by $\boldsymbol{p}_i = [x_i, y_i]^\top \in \mathbb{R}^2,\ i = 1,2$, respectively. Therefore, $f_i(x_i,y_i) = 0,\ i = 1,2$. Two more *linearly independent* equations are required to solve for all four unknown coordinates. The remaining equations are found through a geometric consideration, as explained below.

It is well-known (as the geometric consequence of the Karush-Kuhn-Tucker condition) that the *distance function* $\|\boldsymbol{p}_2 - \boldsymbol{p}_1\|$ attains criticality when the line joining $\boldsymbol{p}_1, \boldsymbol{p}_2$ aligns itself with the *normals* to $\mathcal{C}_i$ at $\boldsymbol{p}_i$ (see Fig. 2). Consequently:

$$\boldsymbol{l} \times \boldsymbol{n}_i = \boldsymbol{0},\ i = 1,2; \quad \text{where } \boldsymbol{l} = \boldsymbol{p}_2 - \boldsymbol{p}_1,\ \text{and } \boldsymbol{n}_i = \nabla_{\{x,y\}} f_i(\boldsymbol{p}_i). \tag{3}$$

Since all the vectors appearing in Eq. (3) are coplanar, only two scalar equations result from it, namely, $f_3(x_1,y_1,x_2,y_2) = 0$ and $f_4(x_1,y_1,x_2,y_2) = 0$, where[2]:

$$\begin{aligned} f_3(x_1,y_1,x_2,y_2) :=& (a_2x_1 + a_3y_1 + a_5)x_2 - (a_1x_1 + a_2y_1 + a_4)y_2 - a_2x_1^2 - \\ & \big((a_3 - a_1)y_1 + a_5\big)x_1 + a_2y_1^2 + a_4y_1 = 0, \end{aligned} \tag{4}$$
$$\begin{aligned} f_4(x_1,y_1,x_2,y_2) :=& (b_2x_2 + b_3y_2 + b_5)x_1 - (b_1x_2 + b_2y_2 + b_4)y_1 - b_2x_2^2 - \\ & \big((b_3 - b_1)y_2 + b_5\big)x_2 + b_2y_2^2 + b_4y_2 = 0. \end{aligned} \tag{5}$$

The critical points, $\boldsymbol{p}_i^*$, satisfy the set of equations $f_j = 0,\ j = 1,\dots,4$.

[2] These equations have been cast explicitly in terms of their constituent monomials to bring forth their algebraic structures, which have been instrumental in developing the solution process (see Sections 2.1, 2.2).

2.2 Solution procedure for determining the critical points

The standard process of solving the equations $f_j = 0$ is to reduce them to a univariate equation in one of the unknowns. There can be multiple choices for the sequence of elimination of variables, as well as the final variable itself. In that sense, the sequence presented below is only one of the possible pathways, which takes into cognisance the algebraic structure of the equations.

Since Eq. (4) is linear in $\{x_2, y_2\}$, one of these two variables may be solved in terms of the other easily. In this work, x_2 is computed thus:

$$x_2 = \frac{(a_1x_1 + a_2y_1 + a_4)y_2 + a_2x_1^2 + \big((a_3 - a_1)y_1 + a_5\big)x_1 - a_2y_1^2 - a_4y_1}{a_2x_1 + a_3y_1 + a_5}. \tag{6}$$

In Eq. (6), it is assumed[3] that the denominator of the RHS does not vanish. When Eq. (6) is used in Eq. (5), the LHS of the latter decomposes into two factors upon simplification, leading to $(y_1 - y_2)g_1(x_1, y_1, y_2) = 0$. If $y_1 = y_2$, then from Eq. (6), $x_2 = x_1$. This situation corresponds to the intersection or tangency of two conics, which are deemed out of the scope of the present work; hence, the solution process proceeds further with $g_1(x_1, y_1, y_2) = 0$. Similarly, replacing $\{x, y\}$ with $\{x_2, y_2\}$, respectively, followed by substituting the expression of x_2 in Eq. (2) results in the equation $g_2(x_1, y_1, y_2) = 0$. The degrees of g_1 and g_2 in $\{x_1, y_1, y_2\}$, are $\{3, 3, 1\}$, and $\{4, 4, 2\}$, respectively[4].

Since g_1 is linear in y_2, an expression for y_2 is obtained by solving $g_1 = 0$. Using the said expression of y_2 in $g_2(x_1, y_1, y_2) = 0$ and simplifying, one obtains: $h_1(x_1, y_1) := (a_2x_1 + a_3y_1 + a_5)^2 h_2(x_1, y_1) = 0$, wherein the first factor in the LHS of the equation may be recognised as the square of the denominator of the RHS of Eq. (6). Since the denominator has been presumed to be non-zero previously, $h_2(x_1, y_1)$ must vanish. Therefore, at this point, two equations remain to be solved, namely: $h_2(x_1, y_1) = 0$, and $f_1(x_1, y_1) = 0$.

It is observed that the degrees of $\{x_1, y_1\}$ in $h_2(x_1, y_1)$ are $\{6, 6\}$, while in $f_1(x_1, y_1) = 0$, these are $\{2, 2\}$. Therefore, a univariate equation in either x_1 or y_1 may be obtained by eliminating the other variable through the computation of a *resultant* of h_2 and f_1 w.r.t. one of these variables. However, it would necessitate the symbolic expansion of the determinant of a polynomial matrix, of size 8×8 (using Sylvester's method) or 6×6 (using Bézout's method). Achieving this is very difficult. Moreover, after solving the final univariate equation, the *suppressed variable* (i.e., the one eliminated in the process of computing the said resultant) needs to be obtained via numerical computation of the *nullspace* of one of the matrices mentioned above, which can be challenging in terms of accuracy. Hence, to circumvent both

[3] The analysis of the degenerate cases is planned in the future. However, all the assumptions are validated after the numerical results are obtained.

[4] The analytical expressions of g_1 and g_2 are omitted here due to want of space.

these difficulties simultaneously, the final elimination is achieved via polynomial division, taking advantage of the quadratic nature of f_1. The process is explained below via the elimination of y_1.

Assuming the coefficient of y_1^2 in f_1, i.e., a_3, to be non-zero, h_2 may be divided by f_1 (both being treated as polynomials in y_1 alone for this step), leading to:

$$h_2(x_1, y_1) := h_3(x_1, y_1) f_1(x_1, y_1) + h_4(x_1, y_1) = 0, \tag{7}$$

where $h_3(x_1, y_1)$ is the quotient in the above division. The remainder, $h_4(x_1, y_1)$, can be *at the most* of degree one in y_1, and hence, it may be expressed as:

$$h_4(x_1, y_1) := r_1(x_1) y_1 + r_2(x_1) = 0. \tag{8}$$

Therefore, y_1 may be found in the *closed form* as: $y_1 = -r_2(x_1)/r_1(x_1)$, assuming $r_1(x_1) \neq 0$. Substitution of this expression of y_1 in the equation $f_1(x_1, y_1) = 0$ results in the following 12-degree univariate polynomial equation in x_1:

$$s(x_1) := a_3^5 s_1(x_1) = 0, \quad \text{where } s_1(x_1) = t_1 x_1^{12} + t_2 x_1^{11} + \cdots + t_{13}. \tag{9}$$

Since $a_3 \neq 0$ (as assumed previously), the univariate $s_1(x_1)$ must vanish. The coefficients $t_k,\ k = 1, \ldots, 13$, have been computed as closed-form symbolic expressions[5] in terms of the original inputs, namely, a_j and $b_j,\ j = 1, \ldots, 6$.

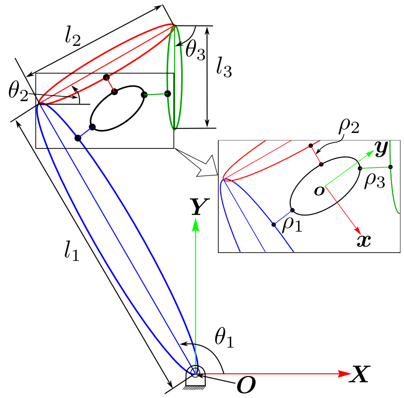
2.3 Computation of the proximity ρ

Equation (9) is solved numerically[6] using the routine "Solve" available in the computer algebra system (CAS), Mathematica [16], with its default numerical precision. All *real* solutions of x_1 (say, n in total) are utilised further to compute the corresponding sets of $\{y_1, y_2, x_2\}$ uniquely. These correspond to the coordinates of the *critical* pairs of points, denoted by $\boldsymbol{p}^*_{i,m}$ on $\mathcal{C}_i$, $i = 1, 2$, respectively, for $m = 1, \ldots, n$. The distances between them are computed as: $d^*_m = \|\boldsymbol{l}^*_m\|$, where $\boldsymbol{l}^*_m = \boldsymbol{p}^*_{2,m} - \boldsymbol{p}^*_{1,m}$. The desired proximity ρ is the *minimum* among the *critical distances*, d^*_m:

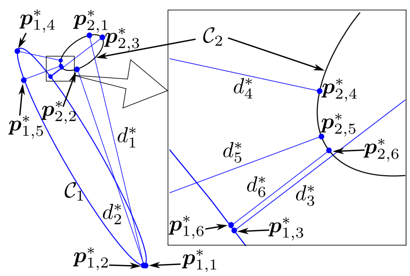
$$\rho = \min(d^*_1, \ldots, d^*_n). \tag{10}$$

[5] The expressions for the coefficients t_k have been shared via GitHub: https://github.com/Bibek67/Coefficients_12_degree_univariate_proximity_ellipses.git, as these are too big to be included here.

[6] It is well-known that roots of polynomials of degrees greater than four with generic coefficients cannot be expressed in terms of radicals.



(a) An example of whole arm manipulation: a planar 3-R arm about to grasp an elliptical object



(b) Six pairs of critical points lying on $\mathcal{C}_1$ and $\mathcal{C}_2$ corresponding to the first link of the serial robot, and the elliptical object it is about to grasp.

Fig. 3: Computation of proximity of pairs of ellipses in the context of whole arm manipulation of an elliptical object by a planar 3-R robot (refer to Table 1 for link-lengths and joint angles pertaining to the configuration shown)

Table 1: Architecture and configuration details of the 3-R robot, and the proximities of the elliptical object from the links of the robot (see Fig. 3a)

Link-length		Joint angle		Proximity	
l_1	1	θ_1	$2\pi/3$	ρ_1	0.0582
l_2	1/2	θ_2	$\pi/6$	ρ_2	0.0529
l_3	1/3	θ_3	$-\pi/2$	ρ_3	0.0750

3 Numerical illustration

As mentioned in Section 1, the only conic relevant to planar robotics is the ellipse (since all the others are not closed and finite). Therefore, the proposed formulation is demonstrated in the context of a planar 3-R robot executing a whole-arm manipulation of an elliptical object, as shown in Fig. 3a. The links of the 3-R robot are modelled as ellipses. The link-lengths[7], and the joint angles, denoted by l_i, and θ_i, respectively, of the 3-R robot are chosen arbitrarily, and these are listed in Table 1. The lengths of the semi-major and semi-minor axes of the ellipse corresponding to the ith link are given by $l_i/2$ and $l_i/2\sqrt{1-e^2}$, respectively, where $e = 0.9900$ is chosen to be the eccentricity of all the ellipses uniformly. The proximities of the elliptical links

[7] All the length parameters are scaled by l_1, thus rendering them unitless. All the angles are expressed in radians, unless stated otherwise.

of the robot to the elliptical object are denoted by ρ_i (see Fig. 3a). The lengths of the semi-minor and semi-major axes of the elliptical object are chosen to be 0.0500 and 0.1000, respectively. The centre of the object is placed at $\boldsymbol{o} = [-0.2500, 0.8500]^\top$, and its major axis is rotated through -0.9369 radians in a CCW manner about the $\boldsymbol{Z}$-axis (see Fig. 3a). With the above settings, the respective equations of the ellipses corresponding to the first link and the object are:

$$
\begin{aligned}
f_1 :=& x^2 + \frac{341369568}{303638161}xy + \frac{10597}{30199}y^2 + \frac{398}{30199}x - \frac{10424282}{456662147}y = 0,\\
f_2 :=& x^2 - \frac{105352484}{75517419}xy + \frac{78011382}{54323053}y^2 + \frac{98213852}{58259009}x\\
&- \frac{137399623}{49245806}y + \frac{63371914}{45537655} = 0.
\end{aligned}
$$

Using the coefficients of the above two ellipses, the univariate in Eq. (9) is solved numerically. Following the procedure presented in Section 2.2, six sets of real solutions are obtained. All the assumptions made in Section 2.2 are numerically verified for each set of real solutions. The corresponding critical points $\boldsymbol{p}^*_{1,m}$ and $\boldsymbol{p}^*_{2,m}$, for $m = 1, \ldots, 6$, and the corresponding distances d^*_m are shown in Fig. 3b. The proximity ρ_1 of the first elliptical link and the object is obtained as (see Fig. 3a): $\rho_1 = d^*_6$, from Eq. (10). In a similar manner, the proximities ρ_2 and ρ_3 between the elliptical object and the second and third elliptical links of the 3-R robot are computed and listed in Table 1. These are depicted visually in Fig. 3a. The values of ρ_i are validated against those obtained using the built-in routine NMinimize of Mathematica (employing the RandomSearch method). This validation is performed by minimising the distance between each pair of ellipses in their parametric forms. It is found that the maximum absolute difference between the results obtained from the proposed method and those obtained from NMinimize does not exceed 2.8163×10^{-6}.

4 Conclusions

This paper presents a new analytical method for the determination of the proximity of two generic (non-intersecting) coplanar conic sections via the solution of a 12-degree univariate polynomial equation. The utility of this method is demonstrated in the context of whole-arm manipulation of an elliptical object using a planar 3-R robot. In future, this work will be extended to include the degenerate/special cases omitted in this work (due to paucity of space).

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