An Algebraic Treatment of Trilaterals

Gökhan Kiper

Abstract Historically geometry was initially handled with distances only, but later on angles and trigonometric functions appeared to model nonlinear systems. Although these latter tools improved the understanding and computation power of humans, today computers work faster with algebraic models. This work presents a dual-vector based formulation for spatial, spherical and planar trilaterals that avoid using trigonometric functions. The problem addressed is finding the location of a vertex line (or point) when the locations of the other two and relative distances/angles of vertices are given. By comparing operations and computation times with classical matrix methods including trigonometric functions it is shown that the pro-posed method proves to be faster. Such an approach has potential to be generalized to other types of trilaterals and other multilaterals that frequently appear in kinematics of mechanisms and robots.

1 Introduction

At the beginning of Book I of Elements [1], Euclid defines *tripleuron*s (trilaterals) in the Definitions section as "figures those which are contained by three straight lines", but then uses the term *trígōnon* (triangle or tricorner) throughout all 13 volumes. According to Boyer [2],

> "*In view of the pre-Hellenic lack of the concept of angle measure, such a study might better be called trilaterometry, or the measure of three sided polygons (trilaterals) than trigonometry, the measure of parts of a triangle. With the Greeks we first find a systematic study of relationships between angles (or arcs) in a circle and the lengths of chords subtending these. … In the works of Euclid there is no trigonometry in the strict sense of the word, but there are theorems equivalent to specific trigonometric laws or formulas.*"

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First works on spherical, and then plane trigonometry started by ancient Greeks for astronomy [3, 4].

The only functions of one variable that can be computed using a finite number of additions, subtractions, and multiplications are polynomials [5]. Other functions need to be computed approximately – with high computational cost. A recent comparison of computational cost of algebraic and transcendental functions is presented in [6]. Although trigonometric functions are essential for several applications today, it is possible to do geometry without them, as in rational trigonometry of Wildberger [7]. Mathematicians such as Husty keep reminding the engineers that "Vector/matrix formulation containing trigonometric functions is arguably the most favored approach used in the engineering research community. A less well-known but nevertheless very successful approach relies on an algebraic formulation." [8]. In this study, we try to devise an intuitive approach for engineers, who are typically familiar with vectors and coordinates. Even though we follow a deductive presentation of the material from spatial triangles to planar and spherical triangles, the formulations are based on the generalization of Proposition 13 in Book II of Elements of Euclid [1].

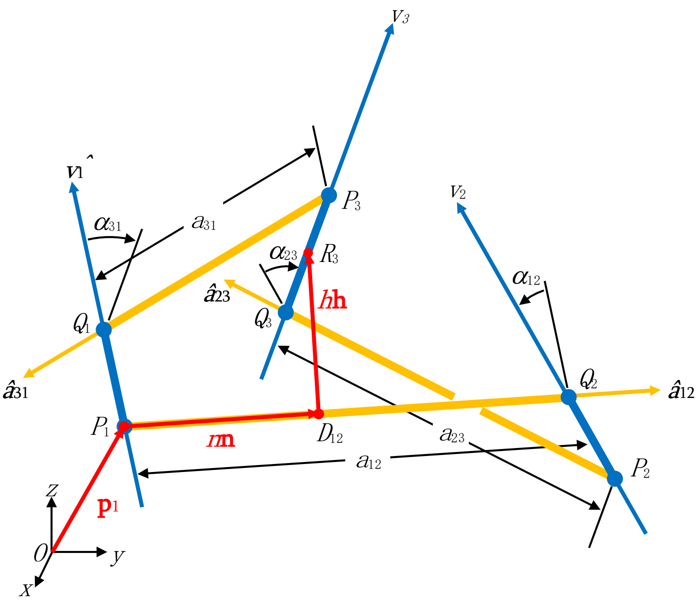

Fig. 1 A spatial trilateral $\hat{\mathbf{v}}_1\hat{\mathbf{v}}_2\hat{\mathbf{v}}_3$

2 Spatial Trilaterals

We follow Yang's [9] definition of spatial trilaterals. For a dual vector $\hat{\mathbf{A}} = \mathbf{u}_A + \varepsilon \mathbf{m}_A$, $\mathbf{u}_A$ is a unit vector along $\hat{\mathbf{A}}$, $\mathbf{m}_A = \mathbf{r}_A \times \mathbf{u}_A$ is moment vector for a position vector $\mathbf{r}_A$ of a point on $\hat{\mathbf{A}}$ with $\varepsilon^2 = 0$. We will make use of reciprocal product (∗), dual scalar product (∘) and dual vector product (⊗) of dual vectors $\hat{\mathbf{A}} = \mathbf{u}_A + \varepsilon \mathbf{m}_A$, $\hat{\mathbf{B}} = \mathbf{u}_B + \varepsilon \mathbf{m}_B$:

$$\hat{\mathbf{A}} * \hat{\mathbf{B}} = \mathbf{u}_A \cdot \mathbf{m}_B + \mathbf{u}_B \cdot \mathbf{m}_A \tag{1}$$

$$\hat{\mathbf{A}} \circ \hat{\mathbf{B}} = (\mathbf{u}_A + \varepsilon \mathbf{m}_A) \circ (\mathbf{u}_B + \varepsilon \mathbf{m}_B) = \mathbf{u}_A \cdot \mathbf{u}_B + \varepsilon (\mathbf{u}_A \cdot \mathbf{m}_B + \mathbf{u}_B \cdot \mathbf{m}_A) = \mathbf{u}_A \cdot \mathbf{u}_B + \varepsilon \hat{\mathbf{A}} * \hat{\mathbf{B}} \tag{2}$$

$$\hat{\mathbf{A}} \otimes \hat{\mathbf{B}} = (\mathbf{u}_A + \varepsilon \mathbf{m}_A) \otimes (\mathbf{u}_B + \varepsilon \mathbf{m}_B) = \mathbf{u}_A \times \mathbf{u}_B + \varepsilon (\mathbf{u}_A \times \mathbf{m}_B + \mathbf{u}_B \times \mathbf{m}_A) \tag{3}$$

$$|\hat{\mathbf{A}}|^2 = \hat{\mathbf{A}} \circ \hat{\mathbf{A}} = (\mathbf{u}_A + \varepsilon \mathbf{m}_A) \circ (\mathbf{u}_A + \varepsilon \mathbf{m}_A) = \mathbf{u}_A \cdot \mathbf{u}_A + 2\varepsilon \mathbf{u}_A \cdot \mathbf{m}_A \tag{4}$$

Consider the spatial trilateral $\hat{\mathbf{v}}_1\hat{\mathbf{v}}_2\hat{\mathbf{v}}_3$ in **Error! Reference source not found.** represented in a Cartesian coordinate frame $O(x, y, z)$ with oriented vertex lines (unit dual vectors) $\hat{\mathbf{v}}_i = \mathbf{u}_i + \varepsilon \mathbf{m}_i$ ($i = 1, 2, 3$). Let P_i be the source foot point and Q_j be the target foot point of the common normal line (edge) ij from $\hat{\mathbf{v}}_i$ to $\hat{\mathbf{v}}_j$ with $j = i+1$ ($j = 1$ if $i = 3$). From primary and dual parts of $\hat{\mathbf{v}}_i \otimes \hat{\mathbf{v}}_j$ vector (See [10]) and unit vector along $\mathbf{P}_i\mathbf{Q}_j$ is

$$\mathbf{n}_{ij} = \begin{cases} \mathbf{u}_i \times \mathbf{u}_j & \text{if } \mathbf{u}_i \times \mathbf{u}_j \neq \mathbf{0} \\ \mathbf{u}_i \times \mathbf{m}_{j-i} & \text{if } \mathbf{u}_i \times \mathbf{u}_j = \mathbf{0} \end{cases}, \quad \mathbf{u}_{ij} = \frac{\mathbf{n}_{ij}}{|\mathbf{n}_{ij}|} \tag{5}$$

where $\mathbf{m}_{j-i} = \mathbf{m}_j - \mathbf{m}_i$. In this definition of $\mathbf{n}_{ij}$ we implicitly assume that $\mathbf{u}_i \times \mathbf{u}_j$ is along the common normal directed from P_i to Q_j. Position vectors of P_i and Q_i are given by

$$\mathbf{p}_i = \mathbf{u}_i \times (\mathbf{u}_{ij} \times \mathbf{m}_{j-i}) \text{ and } \mathbf{q}_i = \mathbf{u}_j \times (\mathbf{u}_{ij} \times \mathbf{m}_{j-i}) \tag{6}$$

A unit dual vector along an edge ij of the trilateral lines is given by

$$\hat{\mathbf{a}}_{ij} = \mathbf{u}_{ij} + \varepsilon \mathbf{m}_{ij} \tag{7}$$

where $\mathbf{m}_{ij} = \mathbf{p}_i \times \mathbf{u}_{ij} = \mathbf{q}_j \times \mathbf{u}_{ij}$ for $i = 1, 2, 3$. The common normal distance is

$$a_{ij} = |\mathbf{P}_i\mathbf{Q}_j| = |\mathbf{q}_j - \mathbf{p}_i| = \mathbf{u}_{ij} \cdot \mathbf{m}_{j-i} = |\hat{\mathbf{v}}_i * \hat{\mathbf{v}}_j| / |\mathbf{n}_{ij}| \tag{8}$$

Let the dual vector of edge ij of the spatial trilateral be

$$\hat{\mathbf{A}}_{ij} = a_{ij}\hat{\mathbf{a}}_{ij} \tag{9}$$

Consider the problem where the poses of two lines $\hat{\mathbf{v}_1} = \mathbf{u}_1 + \varepsilon\mathbf{m}_1$, $\hat{\mathbf{v}_2} = \mathbf{u}_2 + \varepsilon\mathbf{m}_2$ and we seek the pose of a third line $\hat{\mathbf{v}_3} = \mathbf{u}_3 + \varepsilon\mathbf{m}_3$ such that its distances a_{31}, a_{23} to $\hat{\mathbf{v}_1}$, $\hat{\mathbf{v}_2}$, and the angles α_{31}, α_{23} between $\mathbf{u}_3$ & $\mathbf{u}_1$ and $\mathbf{u}_2$ & $\mathbf{u}_3$ are known. For the dual angles $\hat{\alpha_{ij}} = \alpha_{31} + \varepsilon a_{ij}$, $\hat{c}_{ij} = \cos\hat{\alpha_{ij}} = c_{ij} + \varepsilon a_{ij} s_{ij} = \hat{\mathbf{v}_i} \circ \hat{\mathbf{v}_j}$, $\hat{s}_{ij} = \sin\hat{\alpha_{ij}} = s_{ij} - \varepsilon a_{ij} c_{ij} = |\hat{\mathbf{v}_i} \otimes \hat{\mathbf{v}_j}|$ with $c_{ij} = \cos\alpha_{ij} = \mathbf{u}_i \cdot \mathbf{u}_j$, $s_{ij} = \sin\alpha_{ij} = |\mathbf{n}_{ij}|$ (if $\mathbf{n}_{ij}$ is directed from line i to line j) [9]. The primary and dual parts of the following equations define 6 constraints on the 6 components of $\hat{\mathbf{v}_3}$:

$$\hat{\mathbf{v}_1} \circ \hat{\mathbf{v}_3} = \hat{c}_{31} = c_{31} + \varepsilon\, a_{31}\, s_{31}\ ,\ \hat{\mathbf{v}_2} \circ \hat{\mathbf{v}_3} = \hat{c}_{23} = c_{23} + \varepsilon\, a_{23}\, s_{23}\ \text{ and }\ \hat{\mathbf{v}_3} \circ \hat{\mathbf{v}_3} = 1 \tag{10}$$

Let us decompose $\mathbf{u}_3$ as $\mathbf{u}_3 = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_{12}$. Then

$$\begin{matrix} \mathbf{u}_1 \cdot \mathbf{u}_3 = a + b\mathbf{u}_1 \cdot \mathbf{u}_2 = a + bc_{12} = c_{31} \\ \mathbf{u}_2 \cdot \mathbf{u}_3 = a\mathbf{u}_1 \cdot \mathbf{u}_2 + b = ac_{12} + b = c_{23} \end{matrix} \Rightarrow a = \frac{c_{31} - c_{12}c_{23}}{1 - {c_{12}}^2},\ b = \frac{c_{23} - c_{12}c_{31}}{1 - {c_{12}}^2} \tag{11}$$

$$\begin{gathered} \mathbf{u}_3 \cdot \mathbf{u}_3 = |a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_{12}|^2 = a^2 + b^2 + c^2 + 2abc_{12} = 1 \\ \Rightarrow c = \pm\sqrt{1 - a^2 - b^2 - 2abc_{12}} \end{gathered} \tag{12}$$

where the ± sign in the c term corresponds to two mirror solutions in the spherical indicatrix of the spatial trilateral. Due to Eqs. (11-12) we have

$$\mathbf{u}_3 = \frac{c_{31} - c_{12}c_{23}}{1 - {c_{12}}^2}\mathbf{u}_1 + \frac{c_{23} - c_{12}c_{31}}{1 - {c_{12}}^2}\mathbf{u}_2 \pm \sqrt{1 - a^2 - b^2 - 2abc_{12}}\,\mathbf{u}_{12} \tag{13}$$

To compute the moment vector $\mathbf{m}_3$, let's resolve the position vector of a point R_3 on line 3 for some real n and h as

$$\mathbf{r}_3 = \mathbf{p}_1 + n\mathbf{m}_{12} + h\mathbf{h}_3 \text{ with } \mathbf{h}_3 = \mathbf{u}_3 \times \mathbf{n}_{12} \tag{14}$$

Here, $n\mathbf{m}_{12}$ is along edge 12 and $h\mathbf{h}_3$ is the common perpendicular of line 3 and edge 12 with foot point R_3 on line 3. Recall that $\mathbf{p}_1 = \mathbf{OP}_1$ is the foot of edge 12 ($\mathbf{P}_1\mathbf{Q}_2$) on line 1. $n\mathbf{m}_{12} = \mathbf{P}_1\mathbf{D}_{12}$ is the projection of $\mathbf{P}_1\mathbf{R}_3 = \mathbf{r}_3 - \mathbf{p}_1$ on $\mathbf{P}_1\mathbf{Q}_2$ on the planar trilateral $P_1Q_2R_3$ and it can be somehow conceived as the projection of edge 31 onto edge 12 (see Fig. 1). The common normal $h\mathbf{h}_3 = \mathbf{D}_{12}\mathbf{R}_3$ is somehow the altitude from line 3 onto edge 12. To evaluate $\mathbf{P}_1\mathbf{R}_3$, we can use the shortest distances a_{31}, a_{23}. Just as for any two points on lines 3 and 1 or 2, $\mathbf{P}_1\mathbf{R}_3 \cdot \mathbf{n}_{31} = -a_{31}$ s_{31}, and $\mathbf{Q}_2\mathbf{R}_3 \cdot \mathbf{n}_{23} = a_{23}\, s_{23}$:

$$\mathbf{P}_1\mathbf{R}_3 \cdot \mathbf{n}_{31} = (n\mathbf{m}_{12} + h\mathbf{h}_3) \cdot \mathbf{n}_{31} = \mathbf{n}_{12} \cdot \mathbf{n}_{31} n + \mathbf{h}_3 \cdot \mathbf{n}_{31} h = -a_{31} s_{31} \tag{15}$$

$$\mathbf{Q}_2\mathbf{R}_3 \cdot \mathbf{n}_{23} = (-a_{12}\mathbf{u}_{12} + n\mathbf{m}_{12} + h\mathbf{h}_3) \cdot \mathbf{n}_{23} = \mathbf{n}_{12} \cdot \mathbf{n}_{23} n - a_{12}\mathbf{u}_{12} \cdot \mathbf{n}_{23} + \mathbf{h}_3 \cdot \mathbf{n}_{23} h = a_{23} s_{23} \tag{16}$$

Note that $s_{ij} = |\mathbf{n}_{ij}|$, but for computational efficiency it is better to evaluate magnitude of $\mathbf{n}_{ij}$ instead of taking the sine of α_{ij}. Solving Eqs. (15-16) linearly for n and h:

$$n = \frac{-a_{31}s_{31}\mathbf{h}_3 \cdot \mathbf{n}_{23} - \left(a_{23} + a_{12}\mathbf{u}_{12} \cdot \mathbf{u}_{23}\right)s_{23}\mathbf{h}_3 \cdot \mathbf{n}_{31}}{(\mathbf{n}_{12} \cdot \mathbf{n}_{31})(\mathbf{h}_3 \cdot \mathbf{n}_{23}) - (\mathbf{n}_{12} \cdot \mathbf{n}_{23})(\mathbf{h}_3 \cdot \mathbf{n}_{31})} \tag{17}$$

$$h = \frac{\left(a_{23} + a_{12}\mathbf{u}_{12} \cdot \mathbf{u}_{23}\right)s_{23}\left(\mathbf{n}_{12} \cdot \mathbf{n}_{31}\right) + a_{31}s_{31}\left(\mathbf{n}_{12} \cdot \mathbf{n}_{23}\right)}{(\mathbf{n}_{12} \cdot \mathbf{n}_{31})(\mathbf{h}_3 \cdot \mathbf{n}_{23}) - (\mathbf{n}_{12} \cdot \mathbf{n}_{23})(\mathbf{h}_3 \cdot \mathbf{n}_{31})} \tag{18}$$

where $\Delta = (\mathbf{n}_{12} \cdot \mathbf{n}_{31})(\mathbf{h}_3 \cdot \mathbf{n}_{23}) - (\mathbf{n}_{12} \cdot \mathbf{n}_{23})(\mathbf{h}_3 \cdot \mathbf{n}_{31})$. With symbolic manipulations it can be shown that the n solution is not affected with the ± sign in Eq. (13), but the h solution changes its sign. The $(+c, n, +h)$ and $(-c, n, -h)$ solutions for line 3 are on either sides of edge 12. Finally, we have all necessary components to evaluate $\hat{\mathbf{v}}_3$:

$$\mathbf{r}_3 = \mathbf{p}_1 + n\mathbf{m}_{12} + h\mathbf{h}_3 \,,\; \mathbf{m}_3 = \mathbf{r}_3 \times \mathbf{u}_3 \Rightarrow \hat{\mathbf{v}}_3 = \mathbf{u}_3 + \varepsilon\mathbf{m}_3 \tag{19}$$

3 Spherical and Planar Trilaterals

If all vertex lines meet at a point Q, we obtain a spherical trilateral. All foot points P_1, Q_1, P_2, Q_2, P_3, Q_3 of common normal lines are concurrent with Q. The position vector $\mathbf{q} = \mathbf{OQ}$ of the intersection point Q lies on the intersection of the two planes constituted by the origin and one of the lies, hence $\mathbf{q}$ is along the vector product of the normals of the planes, i.e. $\mathbf{m}_1 \times \mathbf{m}_2$. Using $\mathbf{m}_1 = \mathbf{q} \times \mathbf{u}_1$ and $\mathbf{m}_2 = \mathbf{q} \times \mathbf{u}_2$, we can find $\mathbf{q}$ as

$$\mathbf{q} = \frac{\mathbf{m}_1 \times \mathbf{m}_2}{\mathbf{u}_2 \cdot \mathbf{m}_1} \text{ or } \frac{\mathbf{m}_2 \times \mathbf{m}_1}{\mathbf{u}_1 \cdot \mathbf{m}_2} \tag{20}$$

When $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ are given, $\mathbf{u}_3$ can be found using Eq. (13). Since $a_{12} = a_{23} = a_{31} = 0$, $n = h = 0$ by Eqs. (17-18), $\mathbf{r}_3 = \mathbf{q}$ by Eq. (19). $\hat{\mathbf{v}}_3$ is found using Eq. (19).

For a planar trilateral, all vertex lines are parallel to a common direction – without loss of generality, say, z-axis. Then $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}_3 = \mathbf{k}$ (unit vector along z-axis) and all $\alpha_{ij} = 0$. $\mathbf{r}_i$, $a_{ij}\mathbf{u}_{ij} = \mathbf{r}_j - \mathbf{r}_i$, $\mathbf{m}_i = \mathbf{r}_i \times \mathbf{k}$, and $\mathbf{h}_3 = \mathbf{k} \times \mathbf{u}\mathbf{n}_{12}$ are all on the xy-plane. Our problem is to find $\mathbf{p}_3 = \mathbf{q}_3 = \mathbf{r}_3$ for given $\mathbf{p}_1 = \mathbf{q}_1 = \mathbf{r}_1$ and $\mathbf{p}_2 = \mathbf{q}_2 = \mathbf{r}_2$. From Eq. (1)

$$\mathbf{u}_{12} = \frac{\mathbf{k} \times (\mathbf{r}_2 \times \mathbf{k} - \mathbf{r}_1 \times \mathbf{k})}{|\mathbf{k} \times (\mathbf{r}_2 \times \mathbf{k} - \mathbf{r}_1 \times \mathbf{k})|} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{a_{12}}, \ \mathbf{u}_{23} = \frac{\mathbf{r}_3 - \mathbf{r}_2}{a_{23}} \ \text{and} \ \mathbf{u}_{31} = \frac{\mathbf{r}_1 - \mathbf{r}_3}{a_{31}} \tag{21}$$

Let $n' = ns_{12}$ and $h' = hs_{12}$. Substituting $\mathbf{r}_3 = \mathbf{p}_1 + n\mathbf{m}_{12} \pm h\mathbf{h}_3 = \mathbf{r}_1 + n'\mathbf{u}_{12} \pm h'\mathbf{k} \times \mathbf{u}_{12}$ into Eq. (21)

$$\mathbf{u}_{23} = \frac{(n' - a_{12})\mathbf{u}_{12} \pm h'\mathbf{k} \times \mathbf{u}_{12}}{a_{23}} \ \text{and} \ \mathbf{u}_{31} = \frac{-n'\mathbf{u}_{12} \mp h'\mathbf{k} \times \mathbf{u}_{12}}{a_{31}} \tag{22}$$

From Eq. (22)

$$\mathbf{u}_{12} \cdot \mathbf{u}_{23} = \frac{n' - a_{12}}{a_{23}}, \ \mathbf{n}_{12} \cdot \mathbf{n}_{23} = \frac{n' - a_{12}}{a_{23}} s_{12} s_{23} \ \text{and} \ \mathbf{n}_{12} \cdot \mathbf{n}_{31} = \frac{-n'}{a_{31}} s_{12} s_{31} \tag{23}$$

Using Eq. (21) along with $\mathbf{h}_3 = \mathbf{k} \times \mathbf{u}_{12}$ and triple vector identities

$$\begin{aligned} \mathbf{h}_3 \cdot \mathbf{n}_{23} &= (\mathbf{k} \times \mathbf{u}_{12}) \cdot \mathbf{u}_{23} s_{23} = \mathbf{k} \cdot \left(\mathbf{u}_{12} \times \frac{(n' - a_{12})\mathbf{u}_{12} \pm h'\mathbf{k} \times \mathbf{u}_{12}}{a_{23}} \right) s_{23} = \frac{h'}{a_{23}} s_{23} \\ \text{and } \mathbf{h}_3 \cdot \mathbf{n}_{31} &= (\mathbf{k} \times \mathbf{u}_{12}) \cdot \mathbf{u}_{31} s_{31} = \mathbf{k} \cdot \left(\mathbf{u}_{12} \times \frac{-n'\mathbf{u}_{12} \mp h'\mathbf{k} \times \mathbf{u}_{12}}{a_{31}} \right) s_{31} = \frac{-h'}{a_{31}} s_{31} \end{aligned} \tag{24}$$

Substituting Eqs. (23-24) into Eqs. (17-18)

$$n' = \frac{-a_{31}{}^2 + a_{23}{}^2 + n'a_{12} - a_{12}{}^2}{-a_{12}} \Rightarrow n' = \frac{a_{12}{}^2 + a_{31}{}^2 - a_{23}{}^2}{2a_{12}} \tag{25}$$

$$h' = \frac{a_{31}{}^2 - n'^2}{h'} \Rightarrow h' = \sqrt{a_{31}{}^2 - n'^2} \tag{26}$$

Eq. (25) is nothing but Proposition 13 of Book II of Elements. Eqs. (25-26) are sometimes called circle-circle intersection method [11].

4 Computational Aspects

There are various ways to quantify computational cost of evaluating mathematical operations. To present up-to-date representative values, let us have a look at the clocks per element (CPE) values in an Intel® Xeon® Gold 6148 Processor (Table 1).

Table 1. Clocks per element (CPE) values in an Intel® Xeon® Gold 6148 Processor for some elementary mathematical operations [12].

Function	CPE	Function	CPE	Function	CPE
Absolute value	0.25	Cosine	2.95	Arctangent	3.53
Addition	0.26	Sine	3.08	Arctangent2	5.73
Multiplication	0.26	Tangent	3.78	Sqr. root	1.15
Division	1.44	Arccosine	3.68	Exponential	1.53
Square	0.2	Arcsine	3.31	Nat. logarithm	2.42

In order to evaluate the computational cost of the formulations in Sections 2 and 3 let us work on some numerical examples. For the spatial trilateral case consider the inverse kinematics problem of a 2R (R: revolute) serial kinematic chain 1-3-2 with R joints along lines 1 and 3. We want to locate intermediate line 3 for given locations of lines 1 and 2. Locations of lines may change in time, but the relative distance and angles of the axes on the links remain constant. This problem can be classically handled using Denavit-Hartenberg (DH) parameters and homogeneous transformation matrices (See [10]), which results in two solutions (elbow up/down solutions). Alternatively, the problem can be posed as finding the two solutions of $\hat{\mathbf{v}}_3 = \mathbf{u}_2 + \varepsilon \mathbf{m}_2$ for given $\hat{\mathbf{v}}_1 = \mathbf{u}_1 + \varepsilon \mathbf{m}_1$, $\hat{\mathbf{v}}_2 = \mathbf{u}_2 + \varepsilon \mathbf{m}_2$, distances a_{23}, a_{31}, twist angles α_{23}, α_{31} (or projection values c_{23}, s_{23}, c_{31}, s_{31}) and dealt with the dual-vector formulation in Section 2.

To compare the two methods 100 randomly generated set of inputs were computed with a C code run in a Windows operated notebook and average computation times were obtained. The CPE value comparison and average computation times for the benchmarks are listed in Table 2.

The dual vector method seems to be approximately 1.5 times faster than the DH method. Similar benchmarking computations can be performed for spherical and planar quadrilaterals as well.

Table 2. Computational times for dual-vector and DH methods

Operation	Intel CPE	Dual-Vector Method		DH Method	
		Count	CPE	Count	CPE
Arccosine	3.68	0	0.00	2	7.36
Sine, cosine	1.72	0	0.00	4	6.88
Square root	0.66	1	0.66	2	1.32
Division	0.57	4	2.28	4	2.28
Addition / multiplication	0.26	65	16.90	72	18.72
Total CPE		19.84		36.56	
Avg. time		238.5 ns		354.3 ns	

5 Discussions and Conclusions

Although angles and trigonometry are powerful tools to understand and formulate geometry, algebraic manipulation is computationally more efficient. This study is merely an initial step for a formulation for handling spatial, spherical and planar trilaterals. The specific problem addressed here was finding the location of a vertex line when the locations of the other two and relative distances/angles of lines are given. This finds application in mechanism science as the inverse kinematics of a 2R chain. Similar formulations can be devised for RP, PR or PP chains as well (P: prismatic joint). In kinematics of spatial open or closed-loop chains, quadrilaterals, pentalaterals and hexalaterals are also of interest. Further studies may address such multilaterals, in order to deal with complex mechanisms such as manipulators with non-spherical wrists and over-constrained closed-loop linkages. Also, it is necessary to work out the details of choosing correct directions of cross products and signs of sine terms.

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References

1. Heath, T. L.: The Thirteen Books of Euclid's Elements, Vol. I. 2nd edn. Dover, New York (1956).
2. Boyer, C. B.: A History of Mathematics. Wiley, New York
3. Dehn, M.: Mathematics, 200 BC-600 AD, The American Mathematical Monthly 51(3), 149–157 (1944).
4. Duke, D. W.: Hipparchus' eclipse trios and early trigonometry. Centaurus 47, 163–177 (2005).
5. Muller J-M.: Elementary Functions – Algorithms and Implementation. 3rd edn. Birkhauser, New York (2016).
6. Kurik, K. A., Laud, P.: Novel approximations of elementary functions in zero-knowledge proofs. In: Skopik, F., Naessens, V., De Sutter, B. (eds.) Availability, Reliability and Security. ARES 2025. Lecture Notes in Computer Science, vol. 15998, pp. 165-182. Springer, Cham (2025).
7. Wildberger, N. J.: Divine Proportions: Rational Trigonometry to Universal Geometry. Wild Egg Books, sydney (2005).
8. Husty, M. L., Walter, D. R.: Mechanism constraints and singularities—the algebraic formulation. In: Müller, A., Zlatanov, D. (eds.) Singular Configurations of Mechanisms and Manipulators. CISM International Centre for Mechanical Sciences, vol. 589, pp. 101-180 Springer, Cham (2019).