Some Properties of Extracting a Planar Rotation by Quaternion Pseudo-Projection

Mehdi Ghiassi and Andrés Kecskeméthy

Abstract Extracting a single rotational "component" from a 3D rotation about a specified plane normal, termed here "planar rotation", is a recurrent task in many fields, including biomechanics, where clinically interpretable joint angles in the sagittal plane are needed, and parallel kinematic machines, where undesired rotations such as "parasitic rotations" are of interest. Unlike vector spaces, where linear projections provide unique solutions (e.g. orthogonal decompositions), 3D rotations are non-Abelian Lie groups and therefore do not admit linear projections onto subgroups. Current approaches rely on Euler angle decompositions or projections of moving axes, which are the same and show ambiguity and inconsistency. In contrast, we previously introduced a novel simple Quaternion-based algebraic pseudo-projection of 3D rotations, avoiding sequence dependence and inconsistency. This paper further analyzes the relationships between Quaternion pseudo-projection and Euler-angle decompositions. The result is similar to the tilt-and-torsion (T&T) approach, but avoids its singularity problem and also covers the reverse sequence (torsion-and-tilt) automatically.

1 Introduction

Most existing methods for extracting planar rotations either use Euler-angle sequences of elementary rotations, interpreting one elementary rotation as the planar rotation [6], or project body-fixed coordinate axes onto the projection

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plane [4] and use the angles in the plane as the planar rotation. Both show sequence dependence, ambiguity, and distortion [3]. In [5], a review of such methods is presented, showing also that both approaches are identical. As an alternative, we derived a novel Quaternion pseudo-projection method, not depending on decomposition sequence and yielding consistent results. After finalizing these derivations, we found that the result is similar to the tilt-and-torsion (T&T) approach [2], but avoids its singularity problem for vanishing tilt angle and also covers the reverse sequence (torsion-and-tilt) automatically. In this paper, we further analyze the Quaternion pseudo-projection method, unveiling some unexpected properties which can be of use e.g. for navigating close to wrist singularity.

2 Research target and state of the art

Consider a rigid-body frame $\mathcal{K}_E$ rotating relative to a reference frame $\mathcal{K}_0$ (Fig. 1). The question is: what "part" of this rotation can be interpreted to take place about an axis normal to a given plane (here, the z axis), and what is the corresponding "planar" rotation angle α^z? Obviously, due to the nonlinear nature of general spatial rotations, this is a non-trivial task, as there cannot exist a strict projection like in vector spaces. Nevertheless, the need in practice makes a workable approach necessary. This is where the present approach steps in.

Let the rotation from $\mathcal{K}_0$ to $\mathcal{K}_E$ be represented either by an SO(3) parametrization $\mathbf{R}_E$ or, equivalently, by a Quaternion Q by

$$\mathbf{R}_E = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} q_0 \\ \underline{q} \end{bmatrix} = \begin{bmatrix} \cos(\beta/2) \\ \sin(\beta/2)\,\underline{u} \end{bmatrix} \quad \text{with} \quad \underline{q} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}, \tag{1}$$

where q_0 and $\underline{q}$ are the Quaternion real and vector parts, and $\underline{u}$ and β are the unit rotation axis vector and the rotation angle about this axis, respectively, according to Euler's Theorem. The relationship between the two is given as [1]

$$\mathbf{R}(q_0, \underline{q}) = \begin{bmatrix} 1-2\,(\,q_y^2+q_z^2\,) & 2\,(\,-q_0q_z+q_xq_y\,) & 2\,(\,\;q_0q_y+q_xq_z\,) \\ 2\,(\,\;q_0q_z+q_xq_y\,) & 1-2\,(\,q_x^2+q_z^2\,) & 2\,(\,-q_0q_x+q_yq_z\,) \\ 2\,(\,-q_0q_y+q_xq_z\,) & 2\,(\,\;q_0q_x+q_yq_z\,) & 1-2\,(\,q_x^2+q_y^2\,) \end{bmatrix}. \tag{2}$$

From the Quaternion representation, one can recognize two extremal conditions: (a) if the rotation axis $\underline{u}$ is normal to the plane, the Quaternion angle β is identical to the sought "planar" rotation angle α^z, while (b) if the rotation axis $\underline{u}$ lies entirely in the xy plane, then only an in-plane "tilt" rotation takes place, and the "planar" rotation α^z should be zero. While the

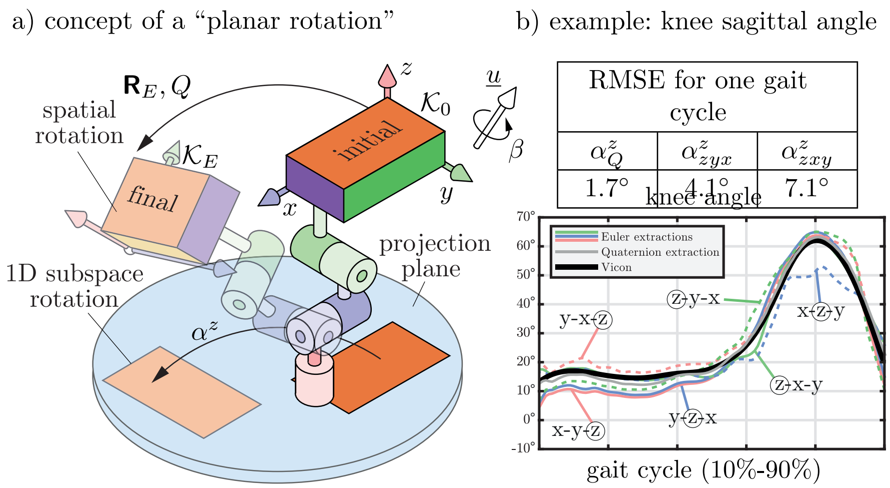

Fig. 1 Example and application of "planar rotation" extraction procedure

first condition is trivial, the second one already shows inconsistencies of the Euler-angle decompositions.

Consider for example the rotation sequence z-y-x with SO(3) representation

$$\mathbf{R}^{\mathrm{zyx}} = \mathrm{Rot}[z, \alpha^z_{zyx}] \cdot \mathrm{Rot}[y, \varphi] \cdot \mathrm{Rot}[x, \psi] = \begin{bmatrix} \cos(\alpha^z_{zyx})\cos(\varphi) & * & * \\ \sin(\alpha^z_{zyx})\cos(\varphi) & * & * \\ * & * & * \end{bmatrix}, \quad (3)$$

where only the two components relevant for determination of the z rotation α^z_{zyx} are represented. If the rotation axis $\underline{u}$ lies in the xy-plane ($q_z = 0$), comparing corresponding coefficients of Eq. (3) and Eq. (2) yields

$$\alpha^z_{zyx} = \mathrm{atan2}\left(\pm 2 q_x q_y \ , \ \pm[1 - 2q_y^2] \right) \neq 0 \qquad \text{for } q_x q_y \neq 0 \quad . \quad (4)$$

Thus one recognizes that the angle α^z_{zyx} is only 0 if the tilt rotation axis is either aligned with the x or the y axis, or in the negligible case that the tilt rotation is very small ($q_x, q_y << 1$), which is again trivial as infinitesimal rotations are commutative and thus elements of a vector space. This shows that the Euler-angle decomposition (as well as the fully equivalent coordinate axis projection) give wrong cues, which also holds for other rotation sequences [5].

An example of these inconsistencies is shown in Fig. 1b for the determination of the knee flexural angle from measurements of IMUs (Inertial Measurement Units) attached to the thigh and shank, respectively [5]. Depending on the Euler angle sequence (colored lines), differing large deviations with respect to the gold standard Vicon (black line) can be observed (the gray line is the result of the Quaternion decomposition further described below). This is due to the spatial rotations of the IMU sensors. Although there is a quest for standardizing the rotation sequence [6], the inconsistencies remain, and the current rotation-sequence standard is just one choice within several inconsistencies to accept. In contrast to this, a proper "planar rotation" decomposition (gray curve), as derived below, can yield much better results, as shown for the corresponding root mean square error in the table of Fig. 1b.

3 Quaternion decomposition and pseudo-projection [5]

As an alternative, consider as basic idea the decomposition of the total rotation Quaternion in two parts [5]: a rotation Q^z about the plane normal, followed by a rotation Q^{xy} about some "tilt" axis in the plane [or vice versa] (Fig. 2)

$$Q \;=\; Q^z \circ Q^{xy} \qquad ; \qquad [\; Q \;=\; Q^{xy} \circ Q^z \;] \tag{5}$$

with both components yet unknown and "$\circ$" denoting Quaternion composition

$$Q = \begin{bmatrix} q_0 \\ \\ \underline{q} \end{bmatrix} = Q^I \circ Q^{II} = \begin{bmatrix} q_0^I q_0^{II} - \underline{q}^I \cdot \underline{q}^{II} \\ \\ q_0^I \underline{q}^{II} + q_0^{II} \underline{q}^I + \underline{q}^I \times \underline{q}^{II} \end{bmatrix} \quad . \tag{6}$$

By definition, the two Quaternion components have the coordinates

$$Q^{\mathrm{z}} = \begin{bmatrix} q_0^{\mathrm{z}} \\ \underline{q}^{\mathrm{z}} \end{bmatrix} = \begin{bmatrix} \cos\left(\alpha_Q^z/2\right) \\ \underline{q}^{\mathrm{z}} \end{bmatrix} \quad \text{with} \quad \underline{q}^{\mathrm{z}} = \begin{bmatrix} 0 \\ 0 \\ q_z^{\mathrm{z}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sin\left(\alpha_Q^z/2\right) \end{bmatrix} \tag{7}$$

$$Q^{\mathrm{xy}} = \begin{bmatrix} q_0^{\mathrm{xy}} \\ \underline{q}^{\mathrm{xy}} \end{bmatrix} \text{ with } \underline{q}^{\mathrm{xy}} = \begin{bmatrix} q_x^{\mathrm{xy}} \\ q_y^{\mathrm{xy}} \\ 0 \end{bmatrix} \qquad \text{and} \qquad \{Q^{xy}\}^{-1} = \begin{bmatrix} q_0^{xy} \\ -\underline{q}^{xy} \end{bmatrix} \tag{8}$$

where α_Q^z is targeted to yield the sought planar rotation angle α^z.

Inserting Eq. (7) and Eq. (8) into the resolved Eq. (5) $Q^z = Q \circ \{Q^{xy}\}^{-1}$ [or $Q^z = \{Q^{xy}\}^{-1} \circ Q$] yields

$$Q^z = \begin{bmatrix} q_0 q_0^{xy} + \underline{q} \cdot \underline{q}^{xy} \\ \\ \\ q_0^{xy}\, \underline{q} - q_0 \underline{q}^{xy} \pm \underline{q} \times \underline{q}^{xy} \\ \\ \end{bmatrix} \left\{ \begin{matrix} = \\ = \\ = \\ = \end{matrix} \right. \begin{bmatrix} q_0 q_0^{xy} + q_x q_x^{xy} + q_y q_y^{xy} \\ q_x q_0^{xy} - q_0 q_x^{xy} \pm q_z q_y^{xy} \\ q_y q_0^{xy} \mp q_z q_x^{xy} - q_0 q_y^{xy} \\ q_z q_0^{xy} \pm q_y q_x^{xy} \mp q_x q_y^{xy} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ = \end{matrix} \begin{bmatrix} q_0^z \\ 0 \\ 0 \\ q_z^z \end{bmatrix} \tag{9}$$

with the upper sign for $Q = Q^z \circ Q^{xy}$ [and the lower one for $Q \;=\; Q^{xy} \circ Q^z$]. From the two middle equations one can resolve for q_x^{xy} and q_y^{xy}, respectively, as

$$q_x^{xy} = \frac{q_0 q_x \pm q_y q_z}{q_0^2 + q_z^2}\, q_0^{xy} \qquad ; \qquad q_y^{xy} = \frac{q_0 q_y \mp q_x q_z}{q_0^2 + q_z^2}\, q_0^{xy} \tag{10}$$

Inserting these into the first and last equation leads to

$$q_0^z = q_0 q_0^{xy} + q_x \frac{q_0 q_x \pm q_y q_z}{q_0^2 + q_z^2}\, q_0^{xy} + q_y \frac{q_0 q_y \mp q_x q_z}{q_0^2 + q_z^2}\, q_0^{xy} \tag{11}$$

$$= \left[1 + \frac{q_x^2 + q_y^2}{q_0^2 + q_z^2}\right] q_0 q_0^{xy} \qquad = \frac{q_0^2 + q_x^2 + q_y^2 + q_z^2}{q_0^2 + q_z^2} q_0 q_0^{xy} \qquad = \frac{q_0}{q_0^2 + q_z^2} q_0^{xy} \tag{12}$$

$$q_z^z = \left[q_z \pm \left\{ q_y \frac{q_0 q_x \pm q_y q_z}{q_0^2 + q_z^2} - q_x \frac{q_0 q_y \mp q_x q_z}{q_0^2 + q_z^2} \right\}\right] q_0^{xy} \tag{13}$$

$$= \left[q_z \pm \frac{\pm(q_y^2 + q_x^2)}{q_0^2 + q_z^2} q_z\right] q_0^{xy} = \left[1 + \frac{(q_y^2 + q_x^2)}{q_0^2 + q_z^2} q_z\right] q_z q_0^{xy} \qquad = \frac{q_z}{q_0^2 + q_z^2} q_0^{xy} \tag{14}$$

With these, the normalizing condition for the Q^z component yields

$$(q_0^z)^2 + (q_z^z)^2 = \frac{(q_0^{xy})^2}{q_0^2 + q_z^2} \stackrel{!}{=} 1 \qquad \Rightarrow \qquad q_0^{xy} = \pm\sqrt{q_0^2 + q_z^2} \ \ , \tag{15}$$

and the solution of the Q^z decomposition is

$$Q^z = \begin{bmatrix} q_0 \\ 0 \\ 0 \\ q_z \end{bmatrix} \frac{\pm 1}{\sqrt{q_0^2 + q_z^2}} \quad \text{and} \quad Q^{xy} = \begin{bmatrix} q_0^2 + q_z^2 \\ q_0 q_x \pm q_y q_z \\ q_0 q_y \mp q_x q_z \\ 0 \end{bmatrix} \frac{\pm 1}{\sqrt{q_0^2 + q_z^2}} \, . \tag{16}$$

where the ± 1 in the numerators of the quotients stem from the square-root solution of Eq. (15), while the $\pm$ signs in the vector part of Q^{xy} reflect the sequence of operation of Q^z and Q^{xy}.

With Eqs. 16 and 7, it follows for the sought planar rotation angle α_Q^z the astonishingly simple solution for both the sequences $Q^z \circ Q^{xy}$ and $Q = Q^{xy} \circ Q^z$

$$\alpha_Q^z = 2 \cdot \text{atan2}\ (\ \pm q_z\ ,\ \pm q_0\ )\,. \tag{17}$$

This means that the obtained planar rotation is just the projection q_z of the spatial Quaternion to the z axis, normalized by the real component q_0 of the spatial Quaternion (thus termed a “pseudo-projection”).

4 Properties of the Quaternion pseudo-projection

First property: The Quaternion pseudo-projection is unique and commutative with respect to the decomposition order $Q^z \circ Q^{xy}$ vs. $Q^{xy} \circ Q^z$.

From Eq. (17) it follows that the angle α_Q^z is independent of the sequence of operation. Moreover, as the atan2 function in Eq. (17) yields solutions separated by 180°, the solution for α_Q^z is unique within 360°. The only singularity occurs when $q_0^2 + q_z^2 = 0$, i.e. when a rotation by 180° happens about a tilt axis in the plane, which can be excluded for cases for which a planar rotation is sought.

Second property: The Quaternion pseudo-projection yields zero planar ("parasitic") rotations when the rotation is about a tilt axis in the plane.

This is trivial, as then $q_z = 0$ and thus $\alpha_Q^z = 0$.

Third property: For a proper Euler-angle sequence with equal first and last traveling axes (e.g. z-x-z), the Q^z "planar" rotation angle is equal to the sum of the first and last Euler angle.

We regard here the case z-x-z, the other cases are similar. Let

$$\mathbf{R}_{zxz} = \mathrm{Rot}[z, \alpha_1] \cdot \mathrm{Rot}[x, \gamma] \cdot \mathrm{Rot}[z, \alpha_2] \tag{18}$$

$$= \begin{bmatrix} \mathrm{c}_{\alpha_1}\mathrm{c}_{\alpha_2} - \mathrm{s}_{\alpha_1}\mathrm{c}_\gamma \mathrm{s}_{\alpha_2} & -\mathrm{c}_{\alpha_1}\mathrm{s}_{\alpha_2} - \mathrm{s}_{\alpha_1}\mathrm{c}_\gamma \mathrm{c}_{\alpha_2} & \mathrm{s}_{\alpha_1}\mathrm{s}_\gamma \\ \mathrm{s}_{\alpha_1}\mathrm{c}_{\alpha_2} + \mathrm{c}_{\alpha_1}\mathrm{c}_\gamma \mathrm{s}_{\alpha_2} & -\mathrm{s}_{\alpha_1}\mathrm{s}_{\alpha_2} + \mathrm{c}_{\alpha_1}\mathrm{c}_\gamma \mathrm{c}_{\alpha_2} & -\mathrm{c}_{\alpha_1}\mathrm{s}_\gamma \\ \mathrm{s}_\gamma \mathrm{s}_{\alpha_2} & \mathrm{s}_\gamma \mathrm{c}_{\alpha_2} & \mathrm{c}_\gamma \end{bmatrix} \tag{19}$$

where s_θ and c_θ denote $\sin(\theta)$ and $\cos(\theta)$ for any angle θ, respectively.

By regarding the trace of $\mathbf{R}$ both in Quaternion form Eq. (2) and in terms of the Euler-angle representation Eq. (19), as well as their skew-symmetric components in the coefficients ρ_{21} and ρ_{12}, one can verify with the shortcut $\bar{\alpha} = \alpha_1 + \alpha_2$:

$$
\begin{aligned}
4q_0^2 \overset{!}{=} \mathrm{Tr}(\mathbf{R})+1 &= \mathrm{c}_{\alpha_1}\mathrm{c}_{\alpha_2} - \mathrm{s}_{\alpha_1}\mathrm{c}_\gamma\mathrm{s}_{\alpha_2} - \mathrm{s}_{\alpha_1}\mathrm{s}_{\alpha_2} + \mathrm{c}_{\alpha_1}\mathrm{c}_\gamma\mathrm{c}_{\alpha_2} + \mathrm{c}_\gamma + 1 && (20)\\
&= \mathrm{c}_{\bar\alpha} + \mathrm{c}_\gamma(1+\mathrm{c}_{\bar\alpha}) + 1 = (1+\mathrm{c}_\gamma)(1+\mathrm{c}_{\bar\alpha}) \quad , && (21)\\
4q_0 q_z \overset{!}{=} \rho_{21} - \rho_{12} &= \mathrm{s}_{\alpha_1}\mathrm{c}_{\alpha_2} + \mathrm{c}_{\alpha_1}\mathrm{c}_\gamma\mathrm{s}_{\alpha_2} + \mathrm{c}_{\alpha_1}\mathrm{s}_{\alpha_2} + \mathrm{s}_{\alpha_1}\mathrm{c}_\gamma\mathrm{c}_{\alpha_2} && (22)\\
&= \mathrm{s}_{\bar\alpha} + \mathrm{c}_\gamma\mathrm{s}_{\bar\alpha} = (1+\mathrm{c}_\gamma)\,\mathrm{s}_{\bar\alpha} \quad . && (23)
\end{aligned}
$$

Thus one obtains for the planar rotation angle according to the Quaternion pseudo-projection Eq. (17)

$$
\tan\left(\frac{\alpha_Q^z}{2}\right) = \frac{q_z}{q_0} = \frac{(1+\mathrm{c}_\gamma)\,\mathrm{s}_{\bar\alpha}}{4q_0^2} = \frac{(1+\mathrm{c}_\gamma)\,\mathrm{s}_{\bar\alpha}}{(1+\mathrm{c}_\gamma)(1+\mathrm{c}_{\bar\alpha})} = \tan\left(\frac{\bar\alpha}{2}\right) \qquad (24)
$$

This is a remarkable property, stating that the proper extraction of a "planar" rotation is the sum of the first and last Euler angle, i.e. neither the first nor the last one.

Fourth property: For a proper Euler-angle sequence as above, the vector part of Q^{xy} is oriented along the axis of the second Euler rotation and rotates by the same angle as the second Euler-angle rotation.

Let the component Q^{xy} be interpreted as a rotation about an axis $\underline{u}^{xy}$ by an angle $\overline{\gamma}$ (Eq. (8)), omitting indices I and II (see below) for better readability

$$
q_0^{xy} = \cos(\overline{\gamma}/2) \quad ; \quad \underline{q}^{xy} = \sin(\overline{\gamma}/2)\,\underline{u}^{xy} \quad ; \quad \underline{u}^{xy} = \left[u_x^{xy}\ ,\ u_y^{xy}\ ,\ 0\right]^{\mathrm{T}} \qquad (25)
$$

Case I: rotation sequence $Q^z \circ Q^{xy}$: With Eqs. (16), (2) and (19), it follows

$$
\begin{aligned}
\pm\sqrt{q_0^2+q_z^2} &= q_0^{xy} && = \cos(\overline{\gamma}/2) && (26)\\
\pm\sqrt{q_0^2+q_z^2}\ q_x^{xy} &= q_0 q_x + q_y q_z = \rho_{32}/2 && = (\mathrm{s}_\gamma \mathrm{c}_{\alpha_2})/2 && (27)\\
\pm\sqrt{q_0^2+q_z^2}\ q_y^{xy} &= q_0 q_y - q_x q_z = \rho_{31}/2 && = -(\mathrm{s}_\gamma \mathrm{s}_{\alpha_2})/2 && (28)
\end{aligned}
$$

Case II: rotation sequence $Q^{xy} \circ Q^z$: Similarly, one obtains

$$
\begin{aligned}
\pm\sqrt{q_0^2+q_z^2}\ q_x^{xy} &= q_0 q_x - q_y q_z = -\rho_{23}/2 && = (\mathrm{s}_\gamma \mathrm{c}_{\alpha_1})/2 && (29)\\
\pm\sqrt{q_0^2+q_z^2}\ q_y^{xy} &= q_0 q_y + q_x q_z = \rho_{13}/2 && = (\mathrm{s}_\gamma \mathrm{s}_{\alpha_1})/2 && (30)
\end{aligned}
$$

Inserting Eq. (26) and Eq. (25) in Eqs. (27) and (28) and Eqs. (29) and (30), respectively, yields for the two component representations of $\underline{u}^{xy}$ (case I / II) :

$$2\,\cos(\overline{\gamma}/2)\,\sin(\overline{\gamma}/2)\ {}^{I}\underline{u}^{xy} \;=\; \sin\overline{\gamma}\ {}^{I}\underline{u}^{xy} \;=\; \sin\gamma\,[\,\cos\alpha_2\,,\ -\sin\alpha_2\,,\ 0\,]^{\mathrm{T}} \tag{31}$$

$$2\,\cos(\overline{\gamma}/2)\,\sin(\overline{\gamma}/2)\,{}^{II}\underline{u}^{xy} \;=\; \sin\overline{\gamma}\,{}^{II}\underline{u}^{xy} \;=\; \sin\gamma\,[\,\cos\alpha_1\,,\quad \sin\alpha_1\,,\ 0\,]^{\mathrm{T}} \tag{32}$$

Considering that the vectors on both sides are unit vectors, it must be $\sin^2\overline{\gamma} = \sin^2\gamma$, hence $\overline{\gamma} = \pm\gamma$, i.e. the rotation angle of the Quaternion component Q^{xy} is equal to the second rotation of the Euler-angle sequence z-x-z up to the sign.

After eliminating $\overline{\gamma}$ and γ from Eq. (31) and Eq. (32), one obtains

$${}^{I}\underline{u}^{xy} = \pm\begin{bmatrix} \cos\alpha_2 & \sin\alpha_2 & 0\\ -\sin\alpha_2 & \cos\alpha_2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} \quad ; \quad {}^{II}\underline{u}^{xy} = \pm\begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0\\ \sin\alpha_1 & \cos\alpha_1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}. \tag{33}$$

As the rotation axis is invariant to the rotation Q^{xy}, the axis ${}^{I}\underline{u}^{xy}$ is given in coordinates of the final frame, while the axis ${}^{II}\underline{u}^{xy}$ is given in coordinates of the initial frame. In both cases, the x axis is rotated to the actual physical axis of the second Euler rotation of the sequence z-x-z, which is invariant to the order of decomposition (case I/II). Fig. 2 shows the corresponding relationships of the Quaternion decomposition and the Euler angles.

In case I, the body-fixed tilt rotation axis $\hat{\underline{q}}^{xy}$ is rotated by $-\alpha_2$ from the body-fixed x axis, and then by the sum $\alpha_1 + \alpha_2$ about the global z axis to its final orientation $\underline{q}^{xy}$, which is the initial x axis rotated by $+\alpha_1$, i.e. the orientation of the second Euler axis. For case II, first one rotates about the final tilt rotation axis $\underline{q}^{xy}$, which is rotated by the angle α_1 from the body-fixed and initial x axis, by γ, and then by the sum angle $\alpha_1 + \alpha_2$ about the rotated z axis, thus yielding again the body-fixed x axis leading ahead of the tilt rotation by α_2. Fig. 2 shows that case II is equivalent to the classical tilt-and-torsion (T&T), i.e. first tilt about an axis in the reference plane and then torsion about the tilted final axis [2]. However, the Quaternion decomposition does not require the determination of the tilt axis ${}^{I}\underline{u}^{xy}$, which becomes singular when the tilt rotation approaches zero, and also automatically covers the inverse sequence (torsion-and-tilt). A further interesting interpretation follows from a precious reviewer remark for which the authors are very grateful: viewing the Quaternion decomposition as a spherical orthogonal triangle with $\underline{q}^{z} \perp \underline{q}^{xy}$, the spherical cosines rule gives $\cos(\beta/2) = \cos(\alpha_Q^z/2)\,\cos(\gamma/2)$.

5 Conclusion

The paper analyzes a novel simple algebraic pseudo-projection Quaternion decomposition approach for extracting a "planar" rotation from a 3D rotation, which is unique and commutative with respect to the order of the

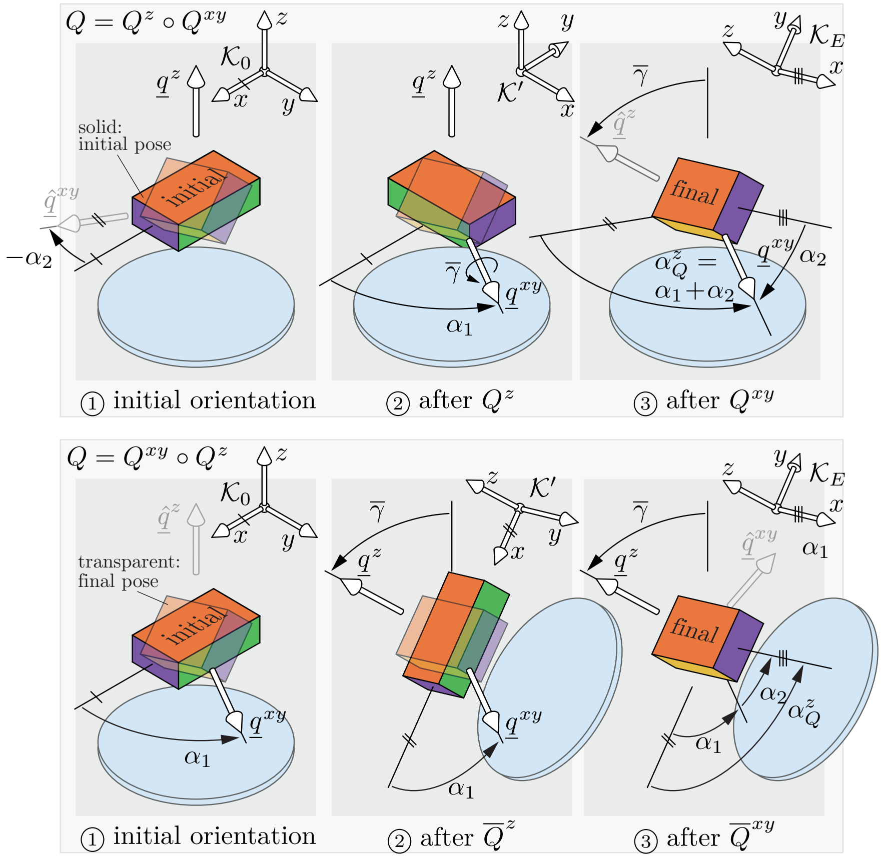

Fig. 2 Visualization of Quaternion decomposition.

decomposition (both tilt-and-torsion and vice-versa). It is shown that (1) the obtained "planar" rotation is the sum of the first and third Euler z-x-z angles, (2) the thus obtained tilt axis corresponds to the physical second Euler axis, and (3) the approach is in contrast to the tilt-and-torsion approach singularity-free even for gimbal-lock passing $\gamma = 0$. This can be of use for navigating a robot through or close to wrist singularity, which is a topic of further research.

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