Dimensionally Consistent Optimal Dynamic Balancing of Slider-Crank Mechanisms Using Equimomental Systems and Screw Theory

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Abstract In this paper, a new procedure for the dynamic balancing of a slider-crank mechanism is presented. First, the inertial properties are represented by equimomental systems of point masses. Then, the acceleration analysis is performed using natural coordinates, which allows us to obtain directly the acceleration of the point masses. The dynamic is solved using the screw theory through Davies' method. A dimensional analysis based on the Buckingham π theorem is introduced to consistently formulate the optimization problem, deriving dimensionless force and torque coefficients that enable the combination of heterogeneous quantities. Finally, the shaking screw is proposed, with which the objective function is formulated using these dimensionless parameters. The proposed procedure is implemented in Matlab®, and the results show a satisfactory dynamic balancing with significant reductions in shaking forces and torques.

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1 Introduction

The notion of equimomental systems, while well established, has been central to the representation of inertial properties of solids and to the dynamic analysis of mechanisms since the nineteenth century. Nevertheless, despite their early introduction, these systems have received limited attention in standard textbooks, and their theoretical implications have remained relatively underexplored. A representative example is the long-standing observation that a minimum of four point masses is required to reproduce the inertial properties of a spatial rigid body, a result that, although known for decades, was only rigorously demonstrated in 2020 [10]. Equimomental point-mass representations offer particular advantages in the dynamic balancing of machines and mechanisms [2, 7], as well as in the reduction of constraint forces and actuation moments [6, 8]. By adopting this representation, the resulting dynamic equations are considerably simplified while simultaneously reducing the number of design parameters involved. Beyond classical mechanical applications, equimomental systems have also been employed in biophysical modeling, as illustrated by the work of Fabián et al. [3], where they are used in the molecular dynamics simulation of cholesterol.

The study of equimomental point-mass systems can be traced back to the nineteenth century [14]. Despite their early origin and clear advantages in the dynamic analysis of mechanisms and machines, this approach has not been widely adopted in contemporary research. Notable progress was achieved by Laus and Selig [10], who developed a rigorous formulation for three-dimensional rigid bodies. Building on these contributions, Núñez et al. [11] proposed a representation of equimomental systems tailored to planar rigid bodies. More recently, related concepts have been applied in celestial mechanics, where the gravitational potential of extended bodies is approximated by means of four point masses positioned at the vertices of an isosceles tetrahedron. This modeling strategy has been successfully employed to approximate the gravitational field of comet 67P/ChuryumovGerasimenko [1].

In the context of dynamic balancing, a critical challenge arises from the need to simultaneously minimize both shaking forces and shaking momentsquantities with fundamentally different physical dimensions. Traditional approaches often rely on ad-hoc weighting factors without rigorous justification for combining these heterogeneous quantities into a single objective function. To address this fundamental issue, the present work incorporates a systematic dimensional analysis based on the Buckingham π theorem. This approach reduces the original eight physical variables into five independent dimensionless groups, among which dimensionless force and torque coefficients are derived. These coefficients provide a physically consistent framework for formulating the multi-objective optimization problem, ensuring that the balancing procedure respects the underlying dimensional homogeneity of the system.

The purpose of this paper is to apply recent advances in the analysis of equimomental systems, combined with dimensional analysis, to a new procedure for the dynamic balancing of a slider-crank mechanism using screw theory. This work is preliminary and will be extended in future publications to increase the scope of the proposed procedure to deal with the balancing of planar and spatial mechanisms in general.

2 Equimomental systems of point masses

This work adopts the formalism introduced by Nuñez et al. [11, 12] to represent equimomental point-mass systems of planar rigid bodies. Furthermore, a simplified representation is proposed for dynamic balancing, using design variables with a geometric perspective, in contrast to [12], where the inertias are used directly. A dimensionally consistent formulation of the optimization problem is also proposed.

Two systems are said to be equimomental when their pseudo-inertia matrices $\widetilde{\boldsymbol{\Xi}}$ are identical. This matrix is diagonalizable and, in the planar case, can be expressed as $\widetilde{\boldsymbol{\Xi}} = m\,\mathrm{diag}\left(a^2, b^2, 1\right)$, where a and b denote the radii of gyration. For simplicity, we choose $a = b = s$ yielding a symmetric planar representation consistent with [11]. The positions of the equimomental point masses are obtained through a non-rigid transformation of the form $\tilde{\mathbf{p}}_i = \widetilde{\mathbf{D}}\tilde{\mathbf{q}}_i$ for $i = 1, 2, 3$, where $\widetilde{\mathbf{D}} = \mathrm{diag}(s, s, 1)$ and $\widetilde{\mathbf{q}}_i$ are homogeneous extended position vectors of not transformed triangle, see Figure 1.

It is now observed that $r = ||\mathbf{p}_i|| = \sqrt{2}s$, and therefore $s = r/\sqrt{2}$. Consequently, the non-rigid transformation matrix $\widetilde{\mathbf{D}}$ can be written as $\widetilde{\mathbf{D}} = \mathrm{diag}(r/\sqrt{2}, r/\sqrt{2}, 1)$, where r is the radius of the circumscribed circle of the right triangle, or equivalently, the radius of gyration with respect to the axis perpendicular to the plane. Finally, the pseudo-inertia matrix can be expressed as $\widetilde{\boldsymbol{\Xi}} = m\,\mathrm{diag}\left(r^2/2, r^2/2, 1\right)$. This simplification in the representation of inertial properties opens the possibility of studying the inertial characteristics of combined bodies without resorting to inertia matrices. Furthermore, the authors believe that this problem may admit an elegant geometric solution, which will be addressed in the extended version of this work.

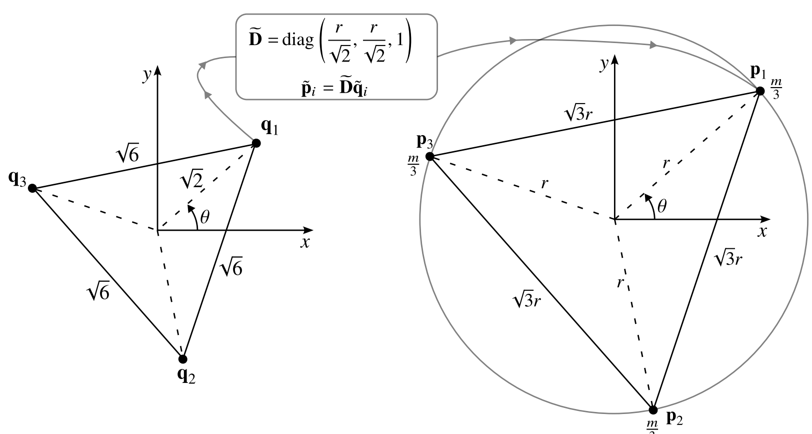

Fig. 1 Representation of an equimomental system using three point masses.

3 Slider-crank mechanism modeling

This section presents an application of equimomental systems to the dynamic balancing of a slidercrank mechanism see Figure 2. The inertial properties of the mechanism are defined by the center of mass and the circumscribed radius of links OB and BC, as well as by the mass of the slider. In this work, we are interested in the redistribution of the mass of links OB and BC; therefore, the design variables are defined as follows:

$$\mathbf{z} = \left[r_1 \; x_{C_1} \; y_{C_1} \; r_2 \; x_{C_2} \; y_{C_2} \right]^T . \tag{1}$$

A systematic formulation for the kinematic and static analysis of planar and spatial mechanisms has been reported in the literature [9]. In the present contribution, this formulation is extended to the inverse dynamics problem through the application of the DAlembert principle [5]. Within this context, inertial effects are modeled through an equivalent set of point masses, allowing the resulting inertia to be expressed in screw form [4]. For the case of equal point masses, this representation leads to the following expression:

$$\$_i^{in} = \begin{bmatrix} \tau_i \\ f_{ix} \\ f_{iy} \end{bmatrix} = -\frac{m_i}{3} \left(\begin{bmatrix} (\mathbf{S}\mathbf{p}_1^i)^T \ddot{\mathbf{p}}_1^i \\ \ddot{\mathbf{p}}_1^i \end{bmatrix} + \begin{bmatrix} (\mathbf{S}\mathbf{p}_2^i)^T \ddot{\mathbf{p}}_2^i \\ \ddot{\mathbf{p}}_2^i \end{bmatrix} + \begin{bmatrix} (\mathbf{S}\mathbf{p}_3^i)^T \ddot{\mathbf{p}}_3^i \\ \ddot{\mathbf{p}}_3^i \end{bmatrix} \right), \tag{2}$$

where, τ_i is inertial torque in the body i, f_{ix} and f_{iy} are the x and y components of the inertial force of body i, $\mathbf{p}_j^i$ for $j = 1, 2, 3.$ is the inertial screw position j of body i, and $\ddot{\mathbf{p}}_j^i$ is the acceleration of point j of body i and $\mathbf{S} = [0 \;\; -1; 1 \;\; 0]$ denotes the 2×2 planar rotation matrix corresponding to a counterclockwise rotation of $\pi/2$ radians.

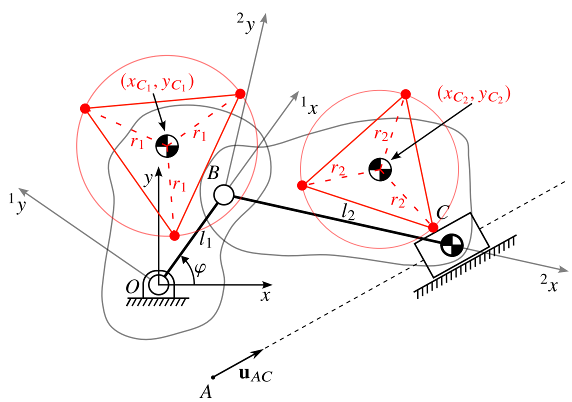

Fig. 2 Slider-crank mechanism. The coordinates of points O and A are (x_O, y_O) and (x_A, y_A), respectively. The piston slides along the line passing through A with direction $\mathbf{u}_{AC} = (x_{\mathbf{u}_{AC}}, y_{\mathbf{u}_{AC}})$.

To determine the accelerations of the point masses, the kinematics of the mechanism is first described by means of constraint equations formulated in natural coordinates. Accordingly, the constraints shown in Fig. 2 can be written as,

$$\mathbf{\Phi}(\mathbf{q}) = \begin{bmatrix} (x_C - x_B)^2 + (y_C - y_B)^2 - l_2^2 \\ (x_C - x_A) y_{\mathbf{u}_{AC}} - (y_C - y_A) x_{\mathbf{u}_{AC}} \\ (x_B - x_O) - l_1 \cos\varphi \\ (y_B - y_O) - l_1 \sin\varphi \end{bmatrix} = \mathbf{0} \tag{3}$$

where, $\mathbf{q} = \begin{bmatrix} x_B \ y_B \ x_C \ y_C \ \varphi \end{bmatrix}^T$. Now, deriving the above equation twice, we obtain

$$\mathbf{\Phi_q}\ddot{\mathbf{q}} = -\dot{\mathbf{\Phi}}_\mathbf{q}\dot{\mathbf{q}} \ , \tag{4}$$

here, $\mathbf{\Phi_q}$ is the Jacobian of the constraint vector Equation (3). Based on this formulation, the shaking loads transmitted to the base can be evaluated following the approach reported in [9] and expressed in compact form through the shaking screw,

$$\$_{sh} = \begin{bmatrix} \boldsymbol{\tau}_{sh} \\ \mathbf{F}_{sh} \end{bmatrix} = -\sum_{i=1}^{n_f} \begin{bmatrix} \mathbf{s}_0^i \times \mathbf{F}_i^* \\ \mathbf{F}_i^* \end{bmatrix} - \sum_{i=1}^{n_t} \begin{bmatrix} \boldsymbol{\tau}_i^* \\ \mathbf{0} \end{bmatrix} \tag{5}$$

where $\mathbf{F}_i^*$ and $\boldsymbol{\tau}_i^*$ are the i-th base reactions, n_f and n_t are their numbers, and $\mathbf{s}_0^i$ is the position vector of $\mathbf{F}_i^*$. The coordinates of the shaking screw have different dimensions; therefore, a dimensional analysis is required to formulate the optimization problem and consistently combine heterogeneous quantities into a single objective function.

3.1 Dimensional Analysis

To generalize the results and reduce the complexity of the parameter space, a dimensional analysis was performed using the Buckingham π theorem. The physical system is governed by eight independent variables ($n = 8$): mass (m), linear displacement (x), angular velocity (ω), linear velocity (v), linear acceleration (a), angular acceleration (α), force (F), and torque (τ).

These variables are described in terms of three fundamental dimensions ($k = 3$): Mass (M), Length (L), and Time (T). According to the theorem, the problem can be reduced to a set of $n - k = 5$ independent dimensionless groups. By selecting mass (m), linear displacement (x), and angular velocity (ω) as the repeating variables to represent the fundamental basis, the derived five dimensionless parameters ($\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5$) are defined. Three of them are basic numbers that can be used in dimensionless graphs in the future. They are: $\Pi_1 = \frac{v}{x\omega}$ (normalized linear velocity), $\Pi_2 = \frac{a}{x\omega^2}$ (normalized linear acceleration), and $\Pi_3 = \frac{\alpha}{\omega^2}$ (normalized angular acceleration). The two remaining numbers have direct application on the optimization problem:

$$\Pi_4 = \frac{F}{mx\omega^2} \qquad \text{(Force Coefficient)} \tag{6}$$

$$\Pi_5 = \frac{\tau}{mx^2\omega^2} \qquad \text{(Torque Coefficient)} \tag{7}$$

With the defined dimensionless coefficients, the optimization problem can be consistently formulated as follows:

$$\begin{aligned} \underset{\mathbf{z}}{\text{minimize}} \quad & f(\mathbf{z}) = \sum_{j=1}^{n} (1-\nu)\Pi_4^2 + \nu\Pi_5^2 \\ \text{subject to} \quad & \underline{\mathbf{z}} \leq \mathbf{z} \leq \bar{\mathbf{z}} \end{aligned} \tag{8}$$

where $j = 1, \ldots, n$ denotes the discrete mechanism states over one full cycle, n is the total number of evaluated states, and $\nu \in [0, 1]$ where $\nu = 0$ is a pure shaking force balancing, $\nu = 1$ is a pure shaking moment balancing. We have adopted in the case studies below $\nu = 0.5$ which is an equally weighted shaking force and shaking moment balancing.

4 Numerical results

Table 1 lists the masses and lengths of the links used in the Matlab® simulations. For simplicity, the piston is assumed to move parallel to the x-axis and to have negligible mass. These assumptions do not entail any loss of generality, as the proposed model can accommodate arbitrary mechanism configurations and account for external forces acting on the piston.

Table 1 Dimensions and masses of links [13].

Body i	1	2	3	**Unit**
Mass m_i	1.64346901	2.51946901	0	[kg]
Length l_i	0.25	0.4	–	[m]

The mechanism is actuated by a motor placed at point O, which operates at a constant angular speed of 500 [rpm]. The limits and initial guesses of the design variables, as well as the corresponding optimal solution, are presented in Table 2. The optimization problem was addressed using the interior-point algorithm available in the Matlab toolbox, with equal weighting assigned to both objective functions ($\nu = 0.5$).

Figure 3(a) compares the shaking torque of the initial mechanism and the balanced mechanism, showing a reduction of 99.99%. Figures 3(b) and 3(c) compare the x- and y-components of the shaking force between the initial and the optimum mechanism. The shaking force in the x-direction was reduced by 65.93%, while the shaking force in the y-direction was reduced by 98.7%.

Table 2 Optimization parameters and results.

Variables	$\underline{\mathbf{z}}$	$\bar{\mathbf{z}}$	$\mathbf{z}_0$	**Optimal**	**Units**
r_1	0	0.780044250948621	0.082151536086865	0.590216198564465	[m]
x_{C_1}	-0.25	0.25	0.125	-0.25	[m]
y_{C_1}	-0.25	0.25	0	-0.000000000458085	[m]
r_2	0	0.630007166350541	0.125142514390283	0.189067535151063	[m]
x_{C_2}	-0.4	0.4	0.2	0.134781419546191	[m]
y_{C_2}	-0.4	0.4	0	-0.000000000512467	[m]

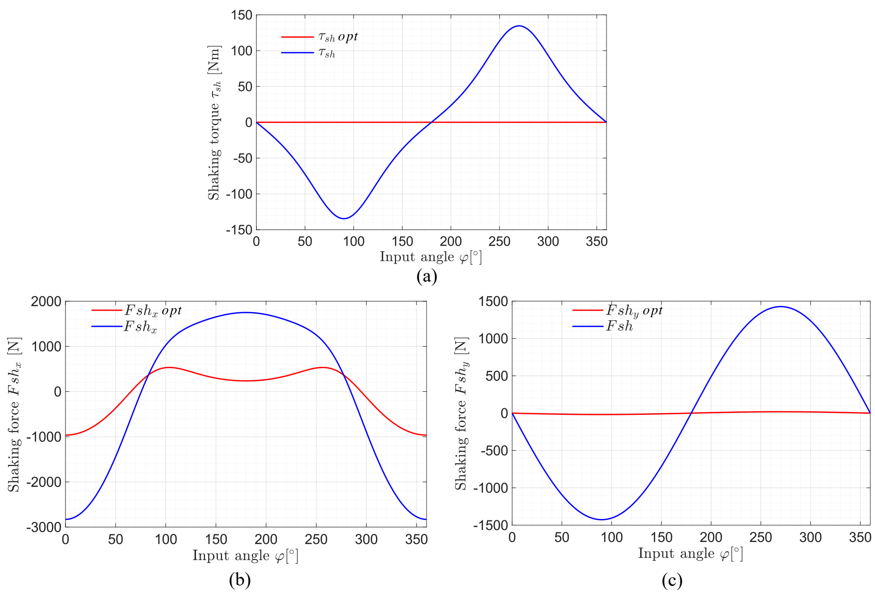

Fig. 3 Comparison between initial solution and optimal balancing of the slider-crank mechanism: (a) shaking torque, (b) x-component of shaking force, and (c) y-component of shaking force.

These results correspond to the idealized Matlab® model adopted in this preliminary study. Therefore, the reported reductions should be interpreted as numerical results for the assumed conditions. The proposed approach redistributes mass through inertial properties while preserving the main mechanism dimensions.

5 Conclusions

A consistent dynamic balancing procedure based on equimomental point-mass systems, screw theory, and dimensional analysis was presented. The use of dimensionless force and torque coefficients enables a physically sound formulation of the optimization problem. The application to a slidercrank mechanism showed substantial reductions in shaking forces and shaking torque. The proposed framework is general in formulation and is demonstrated here for a planar slider-crank mechanism, with extensions left for future work.

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