A Kinetostatic Approach for the Singularity Analysis of a Rehabilitation Parallel Manipulator

Paul Diego, Erik Macho, Francisco J. Campa, Saioa Herrero, Javier Corral, Mikel Diez, and Charles Pinto

Abstract A novel device for balance and gait rehabilitation is presented. It consists of two sensorized footplates driven by two identical 4 degrees-of-freedom parallel manipulators. A simplified kinetostatic model of this manipulator is proposed, which not only reduces the computational cost but also leads to a reduced matrix that enhances the singularity analysis. Using this matrix and placing emphasis on the geometric interpretation of the singularity conditions, a characterization of them is carried out. This characterization has proven useful for dimensional synthesis and for the visual verification of the rehabilitation trajectories defined in the path planning process. A prototype has been developed to validate the usefulness of this device in the rehabilitation field.

1 Introduction

Human balance and gait are fundamental for daily activities such as sitting, standing and walking. When these functions are impaired by neurological disorders, patients experience a significant loss of independence and quality of life [1]. Physical rehabilitation remains the primary treatment to restore these abilities; however, it is time-consuming, and therapists face challenges in delivering repetitive, high-intensity training. In addition, most current assessments rely on subjective and qualitative measures, limiting the ability to personalize rehabilitation programs [2].

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To address these limitations, numerous robotic devices for balance and gait rehabilitation have been developed. Among them, devices based on end-effectors and, namely, those driven by parallel manipulators (PMs) stand out [3], as they combine end-effectors' advantages (i.e., short setup time, easy adaptation to different patients, and versatility to model multiple trajectories) with the excellent properties of PMs. PMs are widely used in the medical field: Birlescu et al. [4] developed PMs for lower limb rehabilitation in acute stage or bedridden patients, Araujo-Gómez et al. [5] introduced a PM for knee diagnosis and rehabilitation, Diego et al. [6] proposed a family of PMs for balance rehabilitation. Campa et al. [2] employed the well-known 3-PRS PM as the driving mechanisms of a balance-assessment device.

The minimum number of degrees-of-freedom (DOF) required for gait and balance rehabilitation is four: two translations (2T) within the patient's sagittal plane and two rotations (2R) about the mediolateral and anteroposterior directions [7]. However, to the authors' knowledge, no PM with this 2T2R motion pattern has been applied in the rehabilitation field. To address this gap, this work introduces BALANCE, a device designed to rehabilitate and objectively assess balance and gait functions in patients with neurological disorders. The system features two independent end-effectors capable of providing controlled and repeatable trajectories. Each end-effector is driven by a novel 2T2R PM, whose singularity analysis is the main contributions of this work.

PMs may reach singular configurations where they gain one or more DOF, compromising both controllability and structural integrity. This gain of DOF has traditionally been associated with the degeneracy of a *Jacobian matrix* derived from the input-output velocity equations [8]. Other researchers have also studied singularities from a static perspective [9], mainly through the *force-transformation matrix*. Although singularities can be identified by evaluating the determinants of these matrices, the resulting expressions are often cumbersome and computationally expensive. To overcome this limitation, we present a *reduced matrix* derived from a simplified kinetostatic analysis. This eases the identification of singularities in the PM used for BALANCE and facilitates the interpretation of the geometric conditions causing them.

2 BALANCE and Driving Parallel Manipulator

The solution proposed for BALANCE device consists of two dynamometric footplates, one for each foot of the patient, which can move synchronously acting as a single tilting base or independently to mimic the trajectory of the feet during gait pattern. Each footplate must have the 4 DOF shown in Fig. 1. That is, two rotational DOF (φ and θ) and two translational DOF (x and z) inside the vertical XZ plane.

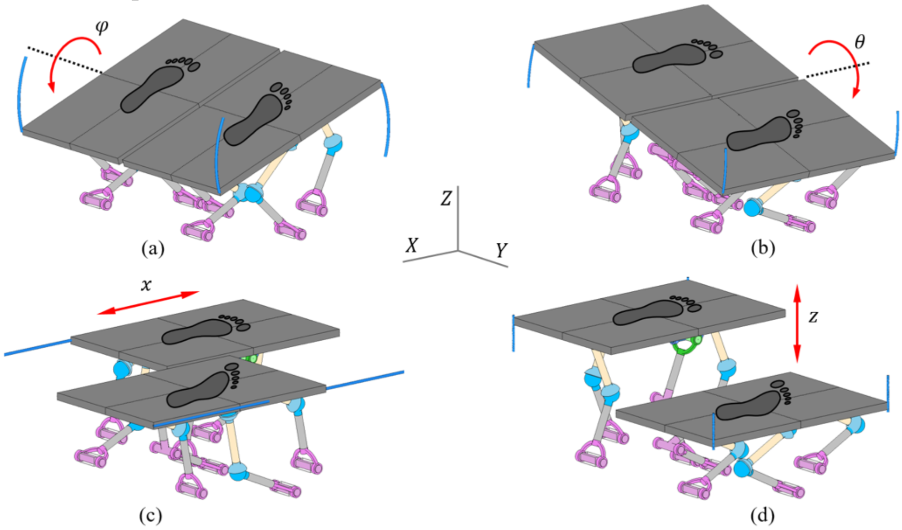


Fig. 1. (a-b) Synchronous rotations (c-d) independent translations along the DOF.

From an anatomical standpoint, φ and θ rotations correspond to the antero-posterior and mediolateral inclinations, and the XZ plane coincides with the patient's sagittal plane. Two PMs arranged beneath each footplate provide these DOF. The schematic representation of one of them is shown in Fig. 2. It consists of a kinematic moving platform connected to a fixed platform through one RRU chain ($i = 1$) and three RSS chains ($i = 2,3,4$). The R joints attached to the fixed platform are the active joints. Regarding mobility, the RRU chain is the only one that constrains the motion space and thus, by itself, defines the DOF of the entire PM and the 2T2R motion pattern. Chains 1 and 3 will be referred to as central chains, while chains 2 and 4 as lateral chains. Let C_i , B_i and A_i denote the position of the active, intermediate and terminal joints.

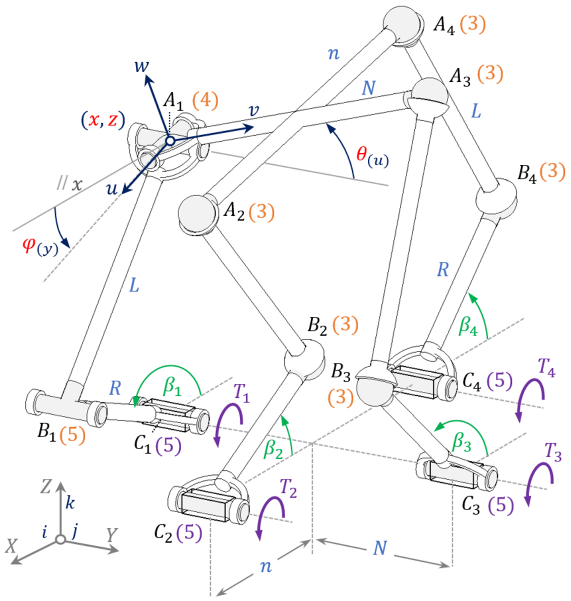


Fig. 2. Schematic of the RRU-3RSS PM with kinematics and kinetostatic variables.

A fixed frame (X, Y, Z) and a moving frame (u, v, w) attached at the center of the U joint (A_1) are considered. The w-axis is normal to the moving platform plane, the v-axis points towards A_3, and u coincides with one of the U joint axes. The position of the moving platform is defined by U joint coordinates (x, z), and its orientation by φ and θ angles, corresponding to rotations about Y-axis $(\boldsymbol{j})$ and u-axis, respectively. Hence, $\boldsymbol{x} = [x, z, \varphi, \theta]$ represents the output variables.

The link connecting C_i and B_i will be referred to as the i-th lower link, and the one connecting B_i and A_i as the i-th upper link. The angles between i-th lower link and X-axis will be denoted by β_i, forming the input vector, $\boldsymbol{q} = [\beta_1, \beta_2, \beta_3, \beta_4]$. For practical reasons, a dimensionally symmetrical PM is used, where R and L denote the lengths of the lower/upper links, and n and N represent half of the central/lateral dimensions of the platforms. Among the 16 working modes, the one shown in Fig.2 is chosen.

3 Kinetostatic Analysis

To select the necessary actuators and properly design the mechanical components, the inverse dynamic problem is solved. Fig. 2 shows the required motor torques $(\boldsymbol{T}_i)$ and, colored in orange, the number of unknowns reactions involved in each kinematic joint. The 47 joints reactions and the 4 motor torques result in a total of 51 unknowns. These can be solved using the classical Newton-Euler approach. Among the 51 unknowns, the 24 corresponding to the active joints reactions and the motor torques (colored in purple) are decoupled, meaning that once the free-body diagrams of the moving platform and the upper links are solved, they can be obtained independently in sequence. Consequently, a subsystem of 51-24 = 27 equations must be solved simultaneously.

Although the general problem has been solved, a simplified approach that enables us to obtain approximate results more efficiently and to identify a reduced matrix useful for singularity analysis is proposed. The simplification assumes that the mass of the upper links is negligible. This is reasonable as the rehabilitation trajectories are carried out under quasi-static conditions and the moving platform, on which the patient stands, is heavier than the links. Besides, unlike the lower links, the upper links are subjected to smaller bending moments, resulting in smaller cross-sections. Under this assumption, SS links only work axially ($\boldsymbol{F_i}$) and the XZ component of the reaction force at the U joint ($\boldsymbol{F_1}$) must be in the direction of RU link.

Accordingly, the number of unknowns in the moving platform is reduced from 13 (Fig.2) to the 6 shown in Fig. 3: the modules of the four forces ($\boldsymbol{F}_i$) in the i-th upper link ($\boldsymbol{s}_i$) direction, the reaction force component at A_1 parallel to Y-axis (F_{A1y}), and the reaction moment value at A_1 ($\boldsymbol{M_{A1}}$), which is normal to the U joint plane ($\boldsymbol{n}_U = \boldsymbol{u} \times \boldsymbol{j}$). The 6 equilibrium equations of the moving platform could be used to solve these unknowns. However, three directions ($\boldsymbol{s}_{1\perp}$, $\boldsymbol{j}$ and $\boldsymbol{u}$) were found for which the equilibrium equations only involve 3 unknowns (F_2, F_3 and F_4), reducing the coupled system to order 3. The unit vector $\boldsymbol{s}_{1\perp}$ is contained in the XZ-plane and is normal to the RU link direction ($\boldsymbol{s}_1$), whereas the subsystem of 3 equations can be set up as follows:

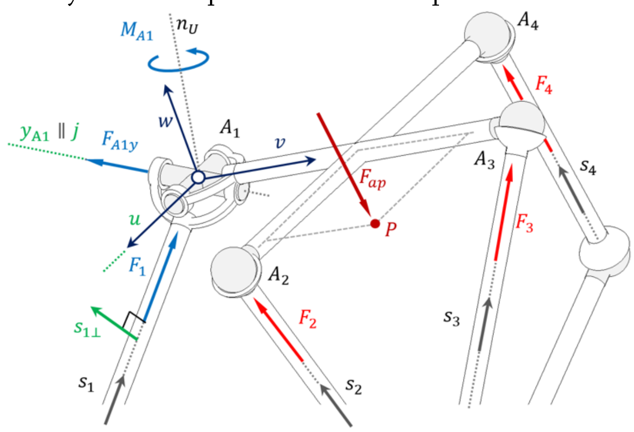


Fig. 3. Free-body diagram of the platform, where F_{ap} is the resultant force applied by the patient.

- **1st equation**. $\boldsymbol{s}_{1\perp}$ is perpendicular to $\boldsymbol{F}_1$ and to F_{A1y} and, thus, the equilibrium of forces in $\boldsymbol{s}_{1\perp}$ direction does not include these two unknowns and yields to Eq. (1):

$$\Sigma F_{s_{1\perp}} = 0 \rightarrow \sum_{i=2}^{4} F_i(\boldsymbol{s}_i{}^T \cdot \boldsymbol{s}_{1\perp}) + \boldsymbol{F}_{ap}{}^T \cdot \boldsymbol{s}_{1\perp} = 0 \tag{1}$$

- **2nd equation**. The axis going through A_1 and being parallel to Y-axis is called y_{A1}. Applying the equilibrium of moments with respect to it leads to Eq. (2):

$$\Sigma N_{y_{A1}} = 0 \rightarrow \sum_{i=2}^{4} F_i(\boldsymbol{A}_1\boldsymbol{A}_i \times \boldsymbol{s}_i)_y + (\boldsymbol{A}_1\boldsymbol{P} \times \boldsymbol{F}_{ap})_y = 0 \tag{2}$$

- **3rd equation**. The equilibrium of moments with respect to $\boldsymbol{u}$ axis leads to Eq. (3):

$$\Sigma N_u = 0 \rightarrow \sum_{i=2}^{4} F_i(\boldsymbol{A}_1\boldsymbol{A}_i \times \boldsymbol{s}_i)_u + (\boldsymbol{A}_1\boldsymbol{P} \times \boldsymbol{F}_{ap})_u = 0 \tag{3}$$

The matrix form of this linear system of equations is:

$$\begin{bmatrix} \boldsymbol{s}_2{}^T \cdot \boldsymbol{s}_{1\perp} & \boldsymbol{s}_3{}^T \cdot \boldsymbol{s}_{1\perp} & \boldsymbol{s}_4{}^T \cdot \boldsymbol{s}_{1\perp} \\ (\boldsymbol{A}_1\boldsymbol{A}_2 \times \boldsymbol{s}_2)^T \cdot \boldsymbol{j} & (\boldsymbol{A}_1\boldsymbol{A}_3 \times \boldsymbol{s}_3)^T \cdot \boldsymbol{j} & (\boldsymbol{A}_1\boldsymbol{A}_4 \times \boldsymbol{s}_4)^T \cdot \boldsymbol{j} \\ (\boldsymbol{A}_1\boldsymbol{A}_2 \times \boldsymbol{s}_2)^T \cdot \boldsymbol{u} & (\boldsymbol{A}_1\boldsymbol{A}_3 \times \boldsymbol{s}_3)^T \cdot \boldsymbol{u} & (\boldsymbol{A}_1\boldsymbol{A}_4 \times \boldsymbol{s}_4)^T \cdot \boldsymbol{u} \end{bmatrix} \begin{bmatrix} F_2 \\ F_3 \\ F_4 \end{bmatrix} = - \begin{bmatrix} \boldsymbol{F}_{ap}{}^T \cdot \boldsymbol{s}_{1\perp} \\ (\boldsymbol{A}_1\boldsymbol{P} \times \boldsymbol{F}_{ap})^T \cdot \boldsymbol{j} \\ (\boldsymbol{A}_1\boldsymbol{P} \times \boldsymbol{F}_{ap})^T \cdot \boldsymbol{u} \end{bmatrix} \tag{4}$$

After solving this system, remaining unknowns in the moving platform (M_{A1}, F_{A1y}, F_1) can be solved independently in sequence. Likewise with B_i and C_i joints unknowns.

4 Singularity Analysis

In addition to reducing the computational cost, the previous approach enables a more direct interpretation of singularity conditions. Due to kinematics-statics duality, analyzing when the coefficient matrix in Eq. (4) becomes singular reveals the poses in which the PM cannot withstand external forces/torques or it gains DOF. As this matrix is only 3×3, its use simplifies the singularity analysis compared with the 6×6 Jacobian matrix derived from the velocity equations. Besides, since all its terms have a geometrical meaning, analyzing when they lead to a null row/column or a linear dependence makes it possible to determine the singularity geometric conditions.

4.1 Null rows and null columns

To have a first row with all null terms, the condition in Eq. (5) must be satisfied for the three SS links ($i = 2,3,4$). From a geometric standpoint this is fulfilled when *all SS links are parallel to the RU link ($\boldsymbol{s}_1$).*

$$\boldsymbol{s}_i{}^T \cdot \boldsymbol{s}_{1\perp} = 0 \iff \boldsymbol{s}_i \perp \boldsymbol{s}_{1\perp} \tag{5}$$

For the second row to be full of null terms, all the SS links must satisfy, at least, one of the three conditions in Eq. (6), where $\forall Y$ is any plane containing Y-axis. This occurs when *the two central SS links* ($i = 2,4$) *are inside the U joint plane (uY).*

$$(\boldsymbol{A}_1\boldsymbol{A}_i \times \boldsymbol{s}_i)^T \cdot \boldsymbol{j} = 0 \iff \begin{cases} \boldsymbol{A}_1\boldsymbol{A}_i \parallel \boldsymbol{j} \text{ or } \boldsymbol{s}_i \parallel \boldsymbol{j} & \text{(6a)} \\ (\boldsymbol{A}_1\boldsymbol{A}_i \times \boldsymbol{s}_i) \perp \boldsymbol{j} \iff \text{both } \boldsymbol{A}_1\boldsymbol{A}_i \text{ and } \boldsymbol{s}_i \in \forall Y \text{ plane} & \text{(6b)} \\ \boldsymbol{A}_1\boldsymbol{A}_i \parallel \boldsymbol{s}_i & \text{(6c)} \end{cases}$$

To have a third row full of null terms, all the SS links must satisfy similar mathematical conditions but replacing $\boldsymbol{j}$ by $\boldsymbol{u}$. From a geometric point of view, this happens whenever all SS links are coplanar with the moving platform plane (uv).

Considering that each column of the coefficient matrix is related to one SS link, the three geometric conditions that lead to null rows, must be fulfilled simultaneously by the same SS link. Hence, to have a null k-th column ($k = 1, 2, 3$), the RU link and the corresponding k-th SS link must be coplanar with the U joint plane.

4.2 Multiple null columns/rows and linear dependence

For the previous geometric conditions, the PM gains one instantaneous DOF, however, it was found that the PM gains 2 instantaneous DOF when *all the SS links are inside U joint plane* and up to 3 DOF when *all the upper links are coplanar to U joint plane*.

The degeneracy of the rank of the reduced matrix also takes place when there is a linear dependence between rows/columns. This dependency arises when *a central upper link becomes parallel to a lateral upper link, and the remaining upper links are parallel to each other*. This condition ruled out the use of some of the working modes.

4.3 Characterization of the Singularity Curves and Path Planning

Breaking the dimensional symmetry would avoid reaching most of the singularities listed before, however, the advantages of symmetry – simpler control, easier assembly, and a reduced manufacturing costs - justify maintaining it. To determine the final link dimensions, the influence of the R/L ratio on the singularity locus was analyzed for 3 cases ($R < L$, $R = L$ and $R > L$) and for specific platform dimensions (n, N). The previous geometric interpretation of singularities allowed us to classify the singularity curves within the workspace (WS) according to the geometric condition causing them.

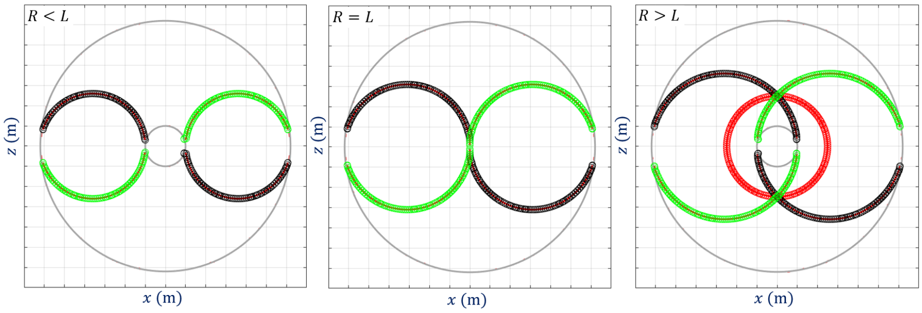


Fig. 4. Singularity curve characterization for different R/L ratio inside the WS for null rotation.

Fig. 4 illustrates an example of this characterization, where three different geometric conditions causing the singularities are identified: lateral upper links lying in U joint plane (black curve – null 3rd column), central upper links lying in U joint plane (green curve – null 2nd row), and parallelism of upper links (red curve – null 1st column).

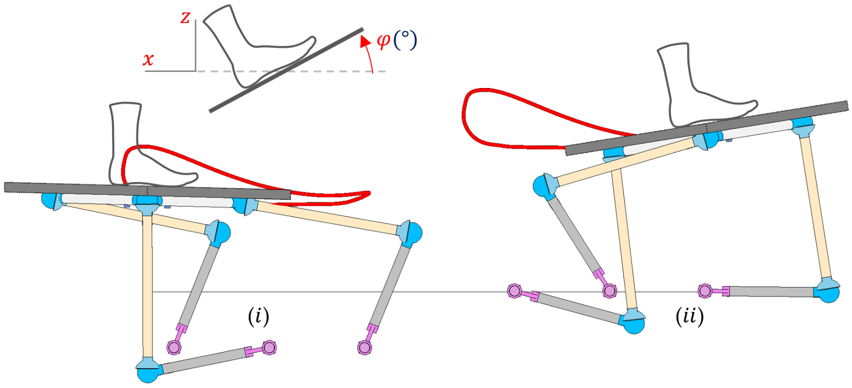
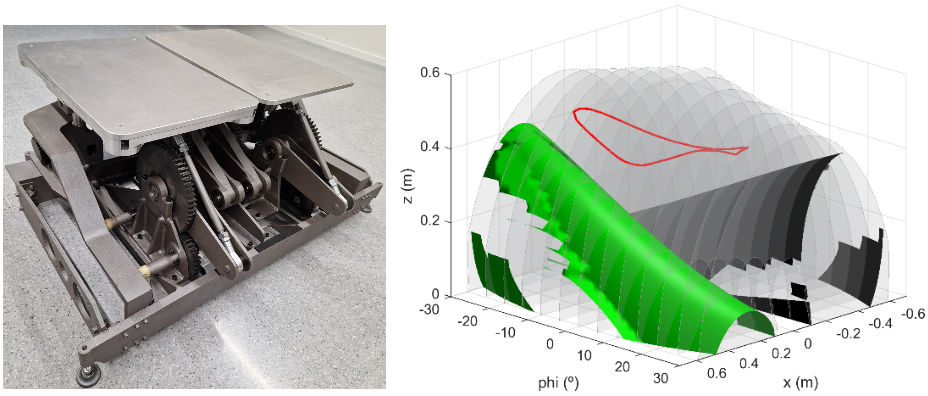


Fig. 5. Critical positions during gait: (i) central (ii) lateral upper links near coplanarity condition.

For the path planning of the rehabilitation exercises, the geometric characterization of the singularities conducted is useful, as it enables a visual inspection of whether the PM approaches a singular configuration along a given trajectory. For instance, along the gait trajectory proposed for BALANCE, the two positions shown in Fig. 5 can be regarded as critical as the (i) central and (ii) lateral upper links are close to be coplanar with U joint plane.

5 Experimental Prototype

A prototype with R = 0.26 m, L = 0.36 m, n = 0.20 m and N = 0.15 m dimensions was developed. These values result from an optimization process aimed at obtaining a WS that allows carrying out the rehabilitation trajectories specified by the medical staff, while meeting kinematic, dynamic and structural requirements. Fig. 6 shows the RRU-3RSS PM during an early assembly stage and the 3D WS for null θ rotation, where the gait trajectory remains within it and does not intersect the singularity surfaces.



(a) (b)

Fig. 6. (a) Prototype of the PM. (b) Gait trajectory and singularity surfaces within the WS.

6 Conclusions

This work introduced the BALANCE device, designed to support the healthcare system by facilitating rehabilitation, reducing the workload of physiotherapists, and providing objective assessment of patient's recovery. The device consists of two sensorized footplates, on which the patient stands, each driven by a novel PM. A simplified kinetostatic model of this PM was proposed, leading to a reduced matrix that enhances both the identification and the geometric interpretation of its singularities. This geometric insight enabled the characterization of the singularity curves that arise within the WS and proved valuable for the path planning process. A prototype was developed and is intended to evaluate its effectiveness in an upcoming study with real patients.

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