Robust Spear Geometry with Plücker Vectors

Sinan Barut and Bertold Bongardt

Abstract Novel robust formulae for describing the pairwise geometry of spears, oriented lines, in three-dimensional space are reported. These rely on spears in terms of Plücker coordinates and account for a unique point representative of each spear. The formulae are in particular useful in computations for robotic systems acting in three-dimensional space.

1 Introduction

The control of a robotic system relies on a precise analysis of the geometry of its kinematic structure. Due to the motions and actions of its involved mechanical pairs, mostly revolute and prismatic joints, the careful study of the geometry of oriented lines in space is required. Thus, practical robot control builds upon the approaches of screw theory [5] and motor calculus [13], with well-known concepts as the product-of-exponentials [17] or the geometric manipulator Jacobian [17]. The study of space geometry in terms of lines is a classic theme that can be traced back to Plücker [15] who introduced six coordinates to specify a line in space. Kotelnikov and Study represented these by 3-vectors of dual numbers and introduced the concept of a dual angle [19]. With the definition of a dual scalar and cross product, von Mises and Strubecker showed how to compute the dual angle, the common orthogonal, and their (dual) product, the motor, between two lines [13, 18].

S. Barut
Gauss Robotics GmbH, 38126 Braunschweig, Germany,
e-mail: sinan.barut@gauss-robotics.de

B. Bongardt
Hochschule Niederrhein UAS, and Forschungsinstitut für Textil und Bekleidung FTB, 41065 Mönchengladbach, Germany
e-mail: bertold.bongardt@hs-niederrhein.de

Details on history and applications of the threads of line geometry and screw theory are given, for example, in [6, 20, 16, 11]. Most central to line geometry is this algebraic localization:

> *"The dual points of the unit sphere are mapped uniquely onto the (real, proper) directed [unit] lines of Euclidean space."* [10]

The transference from real number to dual number expressions, though, is *not* feasible for all geometries:

> *"The given method has to be applied with* ***caution****. The appearance of* ***parallel*** *spears in space induces that the line-geometric theorems need to permit* ***exceptions*** *that do not have analogous in the geometry of the line bundle. These [..] require a dedicated case-wise discussion within an exhaustive analysis."* [19]

In line with this statement, the present paper works out variants of well-known formulae that also function robustly in presence of line pairs with exceptional geometries. The paper is structured as follows: Section 2 introduces to underlying concepts. In Section 3, a set of classic formulas are discussed and their robust versions are developed. Finally, brief conclusions are drawn in Section 4.

2 Prerequistes

Spear: oriented line.

According with Graßman, Plücker, and Study [7, 15, 19, 16], an oriented unit line $\hat{\boldsymbol{\Lambda}}$ is defined by a direction $\hat{\boldsymbol{n}}$ and a moment $\boldsymbol{m}$, where the direction $\hat{\boldsymbol{n}}$ has unit norm, $||\hat{\boldsymbol{n}}|| = 1$, and the moment vector is perpendicular to the direction vector, $\hat{\boldsymbol{n}} \perp \boldsymbol{m}$. In other terms, a unit spear is characterized by

$$\hat{\boldsymbol{n}} * \hat{\boldsymbol{n}} = 1 \qquad\qquad \hat{\boldsymbol{n}} * \boldsymbol{m} = 0 \ ,$$

where the standard inner product of vectors is denoted by $\boldsymbol{a} * \boldsymbol{b} = \sum_i a_i \cdot b_i$. Due to these two conditions, the set of all unit lines forms the tangent bundle $\mathsf{T}(\mathbb{S}^2)$ of the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, also known as the 'Study sphere' [16], and denoted here by $\tilde{\mathbb{S}}^2 = \mathsf{T}(\mathbb{S}^2)$. For the subsequent considerations, the description of a line as a dual vector according to Kotelnikov and Study is advantageous:[1]

$$\hat{\boldsymbol{\Lambda}} = \hat{\boldsymbol{n}} + \epsilon \cdot \boldsymbol{m} \ \in \ \tilde{\mathbb{S}}^2 \subset \tilde{\mathbb{R}}^3 \tag{1}$$

The dual numbers $\tilde{\mathbb{R}} = \mathbb{R} + \epsilon\,\mathbb{R}$ introduce the dual unit ϵ with $\epsilon^2 = 0$ [14, 12].

[1] Alternately, Plücker coordinates are frequently stated in terms of a six-vector of $\mathbb{R}^6$. For further information on Plücker coordinates, consider, for example, [15, 16, 11, 4].

Canonic anchor.

Any unit spear $\hat{\boldsymbol{\Lambda}}$ features a unique point with shortest distance to the origin [2]. This point is called *canonic anchor* $\underline{\boldsymbol{a}}$ here and is obtained via

$$\underline{\boldsymbol{a}} := \hat{\boldsymbol{n}} \times \boldsymbol{m} \; . \tag{2}$$

With this canonic point representative, the interpretation of a unit spear as a vector pair $\hat{\boldsymbol{\Lambda}} \cong (\hat{\boldsymbol{n}}, \boldsymbol{m})$, is augmented to a vector triple $\hat{\boldsymbol{\Lambda}} \cong (\hat{\boldsymbol{n}}, \boldsymbol{m}, \underline{\boldsymbol{a}})$.

Unit screw to motor screw.

In classic screw theory, a motor screw $\boldsymbol{\$}$ transforms an object via a helical motion along a screw axis $\hat{\boldsymbol{\$}}$, an oriented unit line equipped with a particular pitch h: Thus, a screw $\boldsymbol{\$} = \boldsymbol{n} + \epsilon \cdot \check{\boldsymbol{m}}_h$ is obtained as the product of an angle ϕ and a unit screw $\hat{\boldsymbol{\$}} = \hat{\boldsymbol{n}} + \epsilon \cdot \boldsymbol{m}_h$ as

$$\boldsymbol{\$} = \phi \cdot \hat{\boldsymbol{\$}} = \phi \cdot \hat{\boldsymbol{n}} + \epsilon \cdot \phi \cdot \boldsymbol{m}_h \;\in\; \tilde{\mathbb{R}}^3 \; , \tag{3}$$

where the moment $\boldsymbol{m}_h$ of the screw is not required orthogonal to the direction $\hat{\boldsymbol{n}}$. It is obtained from the axis' moment $\boldsymbol{m}$ and a perpendicular component $h \cdot \hat{\boldsymbol{n}}$ along the direction, as $\boldsymbol{m}_h = \boldsymbol{m} + h \cdot \hat{\boldsymbol{n}}$. The scalar product of direction $\hat{\boldsymbol{n}}$ and screw moment $\boldsymbol{m}_h$ defines the pitch $h := \hat{\boldsymbol{n}} * \boldsymbol{m}_h = (\boldsymbol{n} * \check{\boldsymbol{m}}_h)/(\boldsymbol{n} * \boldsymbol{n})$ of the screw which allows to render rotations with $h = 0$, helical motions with $0 < h < \infty$, and translations with $h = \infty$ via $\exp(\boldsymbol{\$}^{\times}) = \mathrm{Ad}(\boldsymbol{D})$ as in (16) in the sequel.

Unit spear to motor screw.

Another way for obtaining a motor could be termed 'cylindrical form' [8], in contrast to the 'helical form' (3): the motor is obtained from two 'free' variables, turn ϕ and shift s, assembled in the dual angle $\tilde{\phi} = \phi + \epsilon \cdot s$. The dual product of $\tilde{\phi}$ and a unit spear $\hat{\boldsymbol{\Lambda}}$ yields a motor $\boldsymbol{\$}$:

$$\boldsymbol{\$} = \tilde{\phi} \cdot \hat{\boldsymbol{\Lambda}} = (\phi + \epsilon \cdot s) \cdot (\hat{\boldsymbol{n}} + \epsilon \cdot \boldsymbol{m}) = \phi \cdot \hat{\boldsymbol{n}} + \epsilon \cdot (\phi \cdot \boldsymbol{m} + s \cdot \hat{\boldsymbol{n}}) \tag{4}$$

In general, the resulting motor $\boldsymbol{\$}$ in (4) is identical with the motor of (3), using the identity $h = s/\phi$. However, the pitch h is defined only implicitly by the combination of free variables, turn and shift, in the cylindrical form.

Polarization. The polarization $\triangle$ exchanges primal component and dual component, the direction $\boldsymbol{n}$ with the moment $\check{\boldsymbol{m}}_h$, of a screw $\boldsymbol{\$}$. It is defined as

$$\triangle \;:\; \tilde{\mathbb{R}}^3 \to \tilde{\mathbb{R}}^3 \;:\; \boldsymbol{\$} \mapsto \triangle(\boldsymbol{\$}) = \triangle(\boldsymbol{n} + \epsilon \cdot \check{\boldsymbol{m}}_h) := \check{\boldsymbol{m}}_h + \epsilon \cdot \boldsymbol{n} \tag{5}$$

Table 1: Overview of screws and spears. The term $\mathbb{S}_0^2$ in row 2 indicates the set of the unit sphere in union with the origin. The tangent bundle of the unit sphere $\mathsf{T}(\mathbb{S}^2)$ in row 4 is also denoted by the Study sphere $\tilde{\mathbb{S}}^2$.

Index	*Concept*	*Symbol*	*Components*	*Special variants*	*Set*	*DoF*
1	motor screw	$\boldsymbol{\$}$	$\boldsymbol{n} + \epsilon\, \mathring{\boldsymbol{m}}_h$	$\boldsymbol{n} + \epsilon\, \boldsymbol{0}$; $\boldsymbol{0} + \epsilon\, \mathring{\boldsymbol{m}}_h$; $\boldsymbol{0} + \epsilon\, \boldsymbol{0}$	$\tilde{\mathbb{R}}^3$	6
2	unit screw	$\hat{\boldsymbol{\$}}$	$\hat{\boldsymbol{n}} + \epsilon\, \boldsymbol{m}_h$	$\hat{\boldsymbol{n}} + \epsilon\, \boldsymbol{0}$; $\boldsymbol{0} + \epsilon\, \mathring{\boldsymbol{m}}_h$	$\mathbb{S}_0^2 + \epsilon\, \mathbb{R}^3$	5
3	spear	$\boldsymbol{\Lambda}$	$\boldsymbol{n} + \epsilon\, \mathring{\boldsymbol{m}}$	$\boldsymbol{n} + \epsilon\, \boldsymbol{0}$	$\mathsf{T}(\mathbb{R}^3_{\neq 0})$	5
4	unit spear	$\hat{\boldsymbol{\Lambda}}$	$\hat{\boldsymbol{n}} + \epsilon\, \boldsymbol{m}$	$\hat{\boldsymbol{n}} + \epsilon\, \boldsymbol{0}$	$\mathsf{T}(\mathbb{S}^2)$	4

For the involutive polarization operator, consider, for example, [10, 19, 17, 4].

Dual normalization. In accordance with [1] for the generic case with $\|\boldsymbol{n}\| \neq 0$, the dual normalization is given by the dual fraction

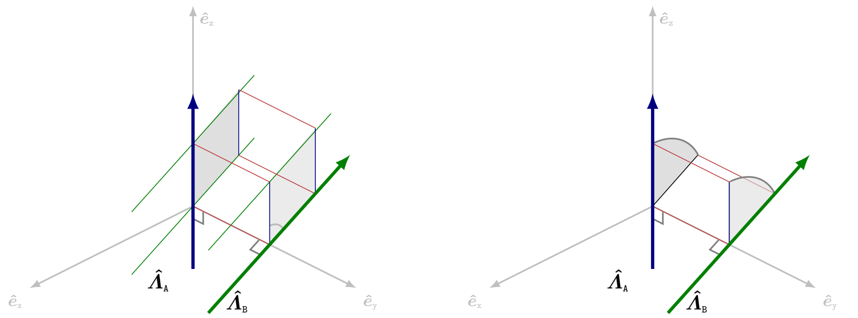
$$\hat{\boldsymbol{\Lambda}} = \widetilde{\mathrm{nrml}}(\boldsymbol{\$}) := \begin{cases} \boldsymbol{\$} \cdot \|\boldsymbol{\$}\|_{\tilde{\mathbb{R}}}^{-1} & \text{if} \quad \|\boldsymbol{n}\| \neq 0 \\ \triangle(\boldsymbol{\$}) \cdot \|\triangle(\boldsymbol{\$})\|_{\tilde{\mathbb{R}}}^{-1} & \text{if} \quad \|\boldsymbol{n}\| = 0 \end{cases} \tag{6}$$

where $\|\boldsymbol{\$}\|_{\tilde{\mathbb{R}}}$ is given by $\|\boldsymbol{\$}\|_{\tilde{\mathbb{R}}} = \|\boldsymbol{n} + \epsilon \cdot \mathring{\boldsymbol{m}}_h\|_{\tilde{\mathbb{R}}} = \|\boldsymbol{n}\| + \epsilon \cdot \frac{\boldsymbol{n} * \mathring{\boldsymbol{m}}_h}{\|\boldsymbol{n}\|} = \phi + \epsilon \cdot s$ and $(\tilde{x})^{-1} = x^{-1} - \epsilon \cdot \mathring{x} \cdot x^{-2}$ is the dual inverse of $\tilde{x} = x + \epsilon \cdot \mathring{x} \in \tilde{\mathbb{R}}$ [14, 1]. The polarization augments for the case of a translator screw with $\|\boldsymbol{n}\| = 0$.

Spears, screws, and motors. An overview of possible direction and moment components for unit spears, spears, unit screws, and motors is given in Table 1.

Table 2: Geometric relationship of spears with respect to their dual angle $\tilde{\vartheta}_{\mathtt{AB}} = \vartheta_{\mathtt{AB}} + \epsilon \cdot s_{\mathtt{AB}}$. For even k the spears are parallel or coincident ($\hat{\boldsymbol{n}}_{\mathtt{A}} = \hat{\boldsymbol{n}}_{\mathtt{B}}$); for odd k the parallelity and coincidence is called anti-parallel or anti-coinciding ($\hat{\boldsymbol{n}}_{\mathtt{A}} = -\hat{\boldsymbol{n}}_{\mathtt{B}}$).

$\tilde{\vartheta}_{\mathtt{AB}}$	$s_{\mathtt{AB}} \neq 0$		$s_{\mathtt{AB}} = 0$		$s_{\mathtt{AB}} \in \mathbb{R}$			
$\vartheta_{\mathtt{AB}} \neq k\pi$	skew	(`S`)	intersecting	(`T`)	regular	(`R`)	$\hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \nparallel \hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$	(11)
$\vartheta_{\mathtt{AB}} = k\pi$	parallel	(`H`)	coincident	(`I`)	coaligned	(`N`)	$\hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \parallel \hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$	(11)
	$\hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \cap \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} = \varnothing$	(25)	$\hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \cap \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} \neq \varnothing$	(25)				



(a) The reciprocal product $\boldsymbol{\Lambda}_{\mathtt{A}} \diamond \boldsymbol{\Lambda}_{\mathtt{B}}$ as the volume $v_{\mathtt{AB}} = (\underline{\boldsymbol{a}}_{\mathtt{A}} - \underline{\boldsymbol{a}}_{\mathtt{B}}) * (\hat{\boldsymbol{n}}_{\mathtt{A}} \times \hat{\boldsymbol{n}}_{\mathtt{B}})$ of the paralletope spanned by the two spears $\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}$ and $\hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$ (7, 8).

(b) The dual angle $\vartheta_{\mathtt{AB}} + \epsilon \cdot s_{\mathtt{AB}}$ of spears as the cylinder segment, obtained via the dual inverse cosine function of the dual inner product $\hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \circledast \hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$ (10, 13).

Fig. 1: Sketches for recipcrocial product and dual angle between unit spears.

3 Geometry of Two Spears

3.1 Classic Formulas

Reciprocal product.

Von Mises [13] introduced the reciprocal product for a pair of lines as a scalar $v_{\mathtt{AB}} \in \mathbb{R}$ defined by

$$v_{\mathtt{AB}} = \hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \diamond \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} := \hat{\boldsymbol{n}}_{\mathtt{A}} * \boldsymbol{m}_{\mathtt{B}} + \boldsymbol{m}_{\mathtt{A}} * \hat{\boldsymbol{n}}_{\mathtt{B}} \; . \tag{7}$$

An inspection with $\boldsymbol{m} = \underline{\boldsymbol{a}} \times \hat{\boldsymbol{n}}$ reveals that $v_{\mathtt{AB}}$ geometrically describes the volume of the parallelotope the two lines span (Figure 1a):

$$v_{\mathtt{AB}} = \hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \diamond \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} = (\underline{\boldsymbol{a}}_{\mathtt{A}} - \underline{\boldsymbol{a}}_{\mathtt{B}}) * (\hat{\boldsymbol{n}}_{\mathtt{A}} \times \hat{\boldsymbol{n}}_{\mathtt{B}}) = -s_{\mathtt{AB}} \cdot \sin(\vartheta_{\mathtt{AB}}) \; . \tag{8}$$

If $\hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \diamond \hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$ vanishes, the two lines are *coplanar*:

$$\hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \diamond \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} = 0 \quad \Leftrightarrow \quad (\hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \parallel \hat{\boldsymbol{\Lambda}}_{\mathtt{B}}) \vee (\hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \cap \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} \neq \varnothing) \; . \tag{9}$$

Coplanar spears are either parallel (intersecting at infinity) or share one (intersecting) or infinite (coincident) common points (Table 2).

Dual inner product.

The dual inner product $\circledast : \tilde{\mathbb{S}}^2 \times \tilde{\mathbb{S}}^2 \to \tilde{\mathbb{R}}$ by Strubecker [18] assigns a dual number $\tilde{c}_{\mathtt{AB}} = \hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \circledast \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} \in \mathbb{D}$ to a pair of spears:

$$\tilde{c}_{\text{AB}} = \hat{\boldsymbol{\Lambda}}_{\text{A}} \circledast \hat{\boldsymbol{\Lambda}}_{\text{B}} = \hat{\boldsymbol{n}}_{\text{A}} * \hat{\boldsymbol{n}}_{\text{B}} + \epsilon \cdot (\hat{\boldsymbol{\Lambda}}_{\text{A}} \diamond \hat{\boldsymbol{\Lambda}}_{\text{B}}) = \cos(\vartheta_{\text{AB}}) - \epsilon \cdot s_{\text{AB}} \cdot \sin(\vartheta_{\text{AB}}) \tag{10}$$

The dual inner product $\tilde{c}_{\text{AB}} = \widetilde{\cos}(\tilde{\vartheta}_{\text{AB}}) = \hat{\boldsymbol{\Lambda}}_{\text{A}} \circledast \hat{\boldsymbol{\Lambda}}_{\text{B}}$ can be read as the dual cosine similarity of spears and as the bilinear form to the Study sphere $\mathsf{T}(\mathbb{S}^2) = \tilde{\mathbb{S}}^2$. The first summand (10) certifies the *alignedness* of two lines $\hat{\boldsymbol{\Lambda}}_{\text{A}}$ and $\hat{\boldsymbol{\Lambda}}_{\text{B}}$ with

$$(\hat{\boldsymbol{n}}_{\text{A}} * \hat{\boldsymbol{n}}_{\text{B}})^2 = 1 \quad \Leftrightarrow \quad \hat{\boldsymbol{\Lambda}}_{\text{A}} \parallel \hat{\boldsymbol{\Lambda}}_{\text{B}} \ . \tag{11}$$

Distance.

The absolute value $|s_{\text{AB}}| \in \mathbb{R}_{\geqslant 0}$ of s_{AB} from (10) between two spears $\hat{\boldsymbol{\Lambda}}_{\text{A}}$ and $\hat{\boldsymbol{\Lambda}}_{\text{B}}$ can be obtained by an equation attributed to Sommerville [9]

$$|s_{\text{AB}}| = \operatorname{dist}(\hat{\boldsymbol{\Lambda}}_{\text{A}}, \hat{\boldsymbol{\Lambda}}_{\text{B}}) := \frac{|\hat{\boldsymbol{\Lambda}}_{\text{A}} \diamond \hat{\boldsymbol{\Lambda}}_{\text{B}}|}{\|\hat{\boldsymbol{n}}_{\text{A}} \times \hat{\boldsymbol{n}}_{\text{B}}\|} \ , \tag{12}$$

which does not hold for the case of coaligned lines (11).

Enclosed dual angle. The dual angle $\tilde{\vartheta}_{\text{AB}} = \vartheta_{\text{AB}} + \epsilon \cdot s_{\text{AB}} \in [0, +\pi] + \epsilon\,\mathbb{R}$ enclosed by $\hat{\boldsymbol{\Lambda}}_{\text{A}}$ and $\hat{\boldsymbol{\Lambda}}_{\text{B}}$ is obtained from $\hat{\boldsymbol{\Lambda}}_{\text{A}} \circledast \hat{\boldsymbol{\Lambda}}_{\text{B}}$ of (10) as

$$\tilde{\vartheta}_{\text{AB}} = \widetilde{\operatorname{acos}}(\hat{\boldsymbol{\Lambda}}_{\text{A}} \circledast \hat{\boldsymbol{\Lambda}}_{\text{B}}) = \operatorname{acos}(\hat{\boldsymbol{n}}_{\text{A}} * \hat{\boldsymbol{n}}_{\text{B}}) - \epsilon \cdot \frac{\hat{\boldsymbol{\Lambda}}_{\text{A}} \diamond \hat{\boldsymbol{\Lambda}}_{\text{B}}}{|\sin(\operatorname{acos}(\hat{\boldsymbol{n}}_{\text{A}} * \hat{\boldsymbol{n}}_{\text{B}}))|} \tag{13}$$

by $\widetilde{\operatorname{acos}}(\tilde{x}) = \operatorname{acos}(x) - \frac{\epsilon \cdot \mathring{x}}{\sqrt{1-x^2}} = \operatorname{acos}(x) - \frac{\epsilon \cdot \mathring{x}}{|\sin(\operatorname{acos}(x))|}$ for $\tilde{x} = x + \epsilon\,\mathring{x} \in \tilde{\mathbb{R}}$. As (12), (13) is invalid for coaligned lines (11) since $|\sin(\operatorname{acos}(\hat{\boldsymbol{n}}_{\text{A}} * \hat{\boldsymbol{n}}_{\text{B}}))| = \|\hat{\boldsymbol{n}}_{\text{A}} \times \hat{\boldsymbol{n}}_{\text{B}}\| = 0$ for $(\hat{\boldsymbol{n}}_{\text{A}} * \hat{\boldsymbol{n}}_{\text{B}})^2 = 1$. Figure 1b illustrates the enclosed dual angle $\tilde{\vartheta}_{\text{AB}}$ as a cylinder segment. The geometric relationship between two spears is classified with respect to their enclosed angle (turn) ϑ_{AB} and distance (shift) s_{AB} in Table 2.

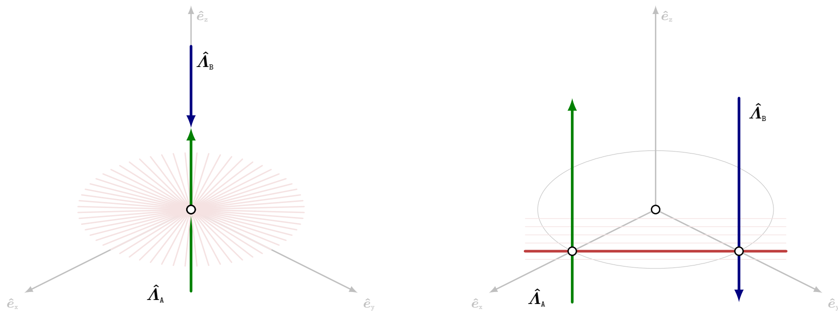
Cross moment and cross product. As a vectorial pendant to the scalar reciprocal product (7), we introduce the 'cross moment' $\check{\boldsymbol{m}}_{\text{AB}} \in \mathbb{R}^3$ by

$$\check{\boldsymbol{m}}_{\text{AB}} = \hat{\boldsymbol{\Lambda}}_{\text{A}} \,\square\, \hat{\boldsymbol{\Lambda}}_{\text{B}} := \hat{\boldsymbol{n}}_{\text{A}} \times \boldsymbol{m}_{\text{B}} + \boldsymbol{m}_{\text{A}} \times \hat{\boldsymbol{n}}_{\text{B}} \ , \tag{14}$$

as a tool to shorten computational expressions. The cross product $\otimes : \tilde{\mathbb{S}}^2 \times \tilde{\mathbb{S}}^2 \to \tilde{\mathbb{R}}^3$ of two spears is defined by

$$\boldsymbol{\$}_{\text{AB}} = \hat{\boldsymbol{\Lambda}}_{\text{A}} \otimes \hat{\boldsymbol{\Lambda}}_{\text{B}} := \hat{\boldsymbol{n}}_{\text{A}} \times \hat{\boldsymbol{n}}_{\text{B}} + \epsilon \cdot (\hat{\boldsymbol{\Lambda}}_{\text{A}} \,\square\, \hat{\boldsymbol{\Lambda}}_{\text{B}}) \tag{15}$$

in formal analogy to the dual inner product (10) with the cross moment (14). For parallel spears with $\hat{\boldsymbol{n}}_{\text{A}} \times \hat{\boldsymbol{n}}_{\text{B}} = \boldsymbol{0}$, this definition of the cross product does not yield a proper unit line. In general, the result is a screw with $\|\boldsymbol{\$}_{\text{AB}}\|_{\tilde{\mathbb{R}}} = |\widetilde{\sin}(\tilde{\vartheta}_{\text{AB}})| = |\sin(\vartheta_{\text{AB}})| + \epsilon \cdot (s_{\text{AB}} \cdot \cos(\vartheta_{\text{AB}}))$ and $(\hat{\boldsymbol{n}}_{\text{A}} \times \hat{\boldsymbol{n}}_{\text{B}}) * \check{\boldsymbol{m}}_{\text{AB}} = \sin(\vartheta_{\text{AB}}) \cdot \cos(\vartheta_{\text{AB}}) \cdot s_{\text{AB}} \neq 0$.



(a) Coincident origin lines have infinite coplanar common orthogonal lines through the origin.

(b) Parallel lines have a unique common orthogonal line that passes the origin plane.

Fig. 2: Common orthogonal lines for particular cases of line geometry.

Algorithm 1 Normalized common orthogonal of two spears $\widetilde{(\text{north})}$

Require: Unit spears $\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}$ and $\hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$ as vector triples $(\hat{\boldsymbol{n}}_{\mathtt{A}}, \boldsymbol{m}_{\mathtt{A}}, \underline{\boldsymbol{a}}_{\mathtt{A}})$ and $(\hat{\boldsymbol{n}}_{\mathtt{B}}, \boldsymbol{m}_{\mathtt{B}}, \underline{\boldsymbol{a}}_{\mathtt{B}})$
Ensure: Orthogonal unit spear $\hat{\boldsymbol{\Lambda}}_{\mathtt{AB}} = \hat{\boldsymbol{n}}_{\mathtt{AB}} + \epsilon \cdot \boldsymbol{m}_{\mathtt{AB}} \in \tilde{\mathbb{S}}^2$ or null

if $(\hat{\boldsymbol{n}}_{\mathtt{A}} * \hat{\boldsymbol{n}}_{\mathtt{B}})^2 = 1$ **and** $\underline{\boldsymbol{a}}_{\mathtt{A}} = \underline{\boldsymbol{a}}_{\mathtt{B}}$ **then** ▷ `I`: coincident lines
  return $\mathbf{0} + \epsilon \cdot \mathbf{0}$ ▷ certificate null $\tilde{\mathbf{0}} \notin \tilde{\mathbb{S}}^2$
else if $(\hat{\boldsymbol{n}}_{\mathtt{A}} * \hat{\boldsymbol{n}}_{\mathtt{B}})^2 = 1$ **then** ▷ `H`: parallel lines
  $\boldsymbol{n}_{\mathtt{AHB}} \leftarrow \hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \,\square\, \hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$ ▷ △ of moment (14)
  $\check{\boldsymbol{m}}_{\mathtt{AHB}} \leftarrow \boldsymbol{m}_{\mathtt{A}} \times \boldsymbol{m}_{\mathtt{B}}$ ▷ △ of direction (15)
  return $\widetilde{\text{nrml}}(\boldsymbol{n}_{\mathtt{AHB}} + \epsilon \cdot \check{\boldsymbol{m}}_{\mathtt{AHB}})$ ▷ normalization (6)
else ▷ `R`: regular lines
  $\boldsymbol{n}_{\mathtt{ARB}} \leftarrow \hat{\boldsymbol{n}}_{\mathtt{A}} \times \hat{\boldsymbol{n}}_{\mathtt{B}}$ ▷ orthogonal direction (15)
  $\check{\boldsymbol{m}}_{\mathtt{ARB}} \leftarrow \hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \,\square\, \hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$ ▷ cross moment (14)
  return $\widetilde{\text{nrml}}(\boldsymbol{n}_{\mathtt{ARB}} + \epsilon \cdot \check{\boldsymbol{m}}_{\mathtt{ARB}})$ ▷ normalization (6)

3.2 Robust Formulae

Motor condition.

An ordered pair of two spears $\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}, \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} \in \tilde{\mathbb{S}}^2$ defines the motor screw $\$_{\mathtt{AB}} = \tilde{\vartheta}_{\mathtt{AB}} \cdot \hat{\boldsymbol{\Lambda}}_{\mathtt{AB}}$ in the 'cylindrical form' [1] of (4). Geometrically, the dual angle $\tilde{\vartheta}_{\mathtt{AB}}$ applied along $\hat{\boldsymbol{\Lambda}}_{\mathtt{AB}}$ transports the spear $\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}$ onto $\hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$, formally as

$$\exp\big(\,\tilde{\vartheta}_{\mathtt{AB}} \cdot \hat{\boldsymbol{\Lambda}}_{\mathtt{AB}}^{\times}\,\big) \cdot \hat{\boldsymbol{\Lambda}}_{\mathtt{A}} = \mathrm{Ad}(\boldsymbol{D}_{\mathtt{AB}}) \cdot \hat{\boldsymbol{\Lambda}}_{\mathtt{A}} = \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} \tag{16}$$

with matrices $\hat{\boldsymbol{\Lambda}}_{\mathtt{AB}}^{\times} = \hat{\boldsymbol{n}}_{\mathtt{AB}}^{\times} + \epsilon \cdot \boldsymbol{m}_{\mathtt{AB}}^{\times} \in \mathrm{ad}(\mathsf{se}(3))$ and $\boldsymbol{v}^{\times} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \in \mathsf{so}(3)$ and $\mathrm{Ad}(\boldsymbol{D}_{\mathtt{AB}}) = \boldsymbol{R}_{\mathtt{AB}} + \epsilon \cdot (\boldsymbol{t}_{\mathtt{AB}}^{\times} \boldsymbol{R}_{\mathtt{AB}}) \in \mathrm{Ad}(\mathsf{SE}(3))$ [1, 3]. For the motor condition in the cylindrical form (16), the orthogonal unit spear $\hat{\boldsymbol{\Lambda}}_{\mathtt{AB}}$ has to be retrieved and $\tilde{\vartheta}_{\mathtt{AB}} \in [0, +\pi] + \epsilon\,\mathbb{R}$ of (13) has to be determined consistently with $\hat{\boldsymbol{\Lambda}}_{\mathtt{AB}}$.

Robust common orthogonal. Using the polarization (5), the cross product (15) is augmented for the case of parallel lines (Figure 2b, Figure 3) with:

$$\boldsymbol{\$}_{\mathtt{AB}} = \chi_{\mathtt{A}}(\hat{\boldsymbol{\Lambda}}_{\mathtt{B}}) := \begin{cases} \hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \otimes \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} & \text{if } \hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \nparallel \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} \\ \triangle\big(\triangle(\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}) \otimes \triangle(\hat{\boldsymbol{\Lambda}}_{\mathtt{B}})\big) & \text{if } \hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \parallel \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} \end{cases} \tag{17}$$

The normalization (6) retrieves the unit spear from the screw $\boldsymbol{\$}_{\mathtt{AB}} = \chi_{\mathtt{A}}(\hat{\boldsymbol{\Lambda}}_{\mathtt{B}})$ via

$$\hat{\boldsymbol{\Lambda}}_{\mathtt{AB}} = \hat{\boldsymbol{n}}_{\mathtt{AB}} + \epsilon \cdot \boldsymbol{m}_{\mathtt{AB}} := \widetilde{\mathrm{nrml}}(\chi_{\mathtt{A}}(\hat{\boldsymbol{\Lambda}}_{\mathtt{B}})) \ . \tag{18}$$

Steps (17) and (18) are rendered in Algorithm 1 as the procedure $\widetilde{\mathrm{north}}(\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}, \hat{\boldsymbol{\Lambda}}_{\mathtt{B}})$ to compute the normalized common orthogonal robustly. The unsolvable case of coincident lines (Figure 2a) is caught first and an infeasible null spear is returned.

Robust distance. The distance $s_{\mathtt{AB}}$ of (12) is redefined in a robust manner, by means of the canonic anchors $\underline{\boldsymbol{a}}_{\mathtt{A}}$ and $\underline{\boldsymbol{a}}_{\mathtt{B}}$ of (2) and their difference vector

$$\underline{\boldsymbol{d}}_{\mathtt{AB}} := \underline{\boldsymbol{a}}_{\mathtt{B}} - \underline{\boldsymbol{a}}_{\mathtt{A}} = (\hat{\boldsymbol{n}}_{\mathtt{B}} \times \boldsymbol{m}_{\mathtt{B}}) - (\hat{\boldsymbol{n}}_{\mathtt{A}} \times \boldsymbol{m}_{\mathtt{A}}) \ , \tag{19}$$

to $\underline{s}_{\mathtt{AB}}$, aligned with the orientation of $\hat{\boldsymbol{n}}_{\mathtt{AB}}$ from (18), via

$$\underline{s}_{\mathtt{AB}} = \mathrm{dist}(\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}, \hat{\boldsymbol{\Lambda}}_{\mathtt{B}}) := \underline{\boldsymbol{d}}_{\mathtt{AB}} * \hat{\boldsymbol{n}}_{\mathtt{AB}} \ . \tag{20}$$

Robust enclosed dual angle.

The enclosed dual angle is obtained robustly via (20) by the dual bivariate function $\widetilde{\mathrm{ang}} : \tilde{\mathbb{S}}^2 \times \tilde{\mathbb{S}}^2 \to [0,\pi] + \epsilon\,\mathbb{R}$ defined as

$$\underline{\tilde{\vartheta}}_{\mathtt{AB}} = \widetilde{\mathrm{ang}}(\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}, \hat{\boldsymbol{\Lambda}}_{\mathtt{B}}) := \mathrm{acos}(\hat{\boldsymbol{n}}_{\mathtt{A}} * \hat{\boldsymbol{n}}_{\mathtt{B}}) + \epsilon \cdot (\underline{\boldsymbol{d}}_{\mathtt{AB}} * \hat{\boldsymbol{n}}_{\mathtt{AB}}) \ . \tag{21}$$

Alternately, the enclosed dual angle $\underline{\tilde{\vartheta}}_{\mathtt{AB}}$ is obtained robustly by combining $\hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \circledast \hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$ from (10) with the dual norm of $\hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \otimes \hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$ from (15) as the arguments

$$\underline{\tilde{\vartheta}}_{\mathtt{AB}} = \widetilde{\mathrm{atan2}}\big(\|\hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \otimes \hat{\boldsymbol{\Lambda}}_{\mathtt{B}}\|_{\tilde{\mathbb{R}}},\ \hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \circledast \hat{\boldsymbol{\Lambda}}_{\mathtt{B}}\big) \tag{22}$$

of the dual arcus function $\widetilde{\mathrm{atan2}}(\tilde{y}, \tilde{x}) = \mathrm{atan2}(y, x) + \epsilon\, \frac{x\mathring{y} - \mathring{x}y}{x^2+y^2}$ for $\tilde{x} = x + \epsilon\, \mathring{x} \in \tilde{\mathbb{R}}$.

Robust foot and intersection points. The closest point on a spear $\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}$ to a spear $\hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$ is denoted as the foot point $\boldsymbol{c}_{\mathring{\mathtt{A}}\mathtt{B}} \in \mathbb{R}^3$ here. Geometrically, a foot point for a spear pair is given as intersection of one spear and the common orthogonal:

$$\boldsymbol{c}_{\mathring{\mathtt{A}}\mathtt{B}} = \pi^{\circ}_{\mathtt{A}}(\hat{\boldsymbol{\Lambda}}_{\mathtt{B}}) = \hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \cap \hat{\boldsymbol{\Lambda}}_{\mathtt{AB}} \qquad \boldsymbol{c}_{\mathtt{A}\mathring{\mathtt{B}}} = \pi^{\circ}_{\mathtt{B}}(\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}) = \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} \cap \hat{\boldsymbol{\Lambda}}_{\mathtt{AB}} \tag{23}$$

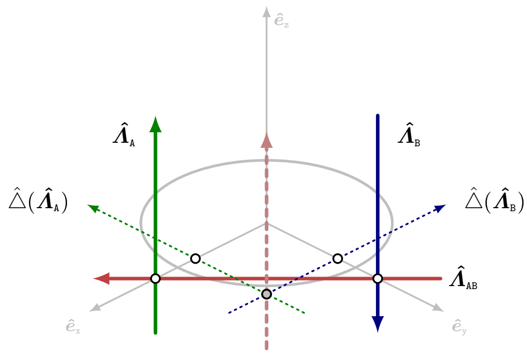


Fig. 3: The common orthogonal $\hat{\boldsymbol{\Lambda}}_{\mathtt{AB}}$ of two (anti-)parallel spears $\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}$ and $\hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$ (see Figure 2b) is obtained with $\hat{\triangle}(\boldsymbol{\$}) \coloneqq \widetilde{\mathrm{nrml}}(\triangle(\boldsymbol{\$}))$, as composition of $\triangle$ in (5) and $\widetilde{\mathrm{nrml}}$ in (6). The polars $\hat{\triangle}(\hat{\boldsymbol{\Lambda}}_{\mathtt{A}})$ and $\hat{\triangle}(\hat{\boldsymbol{\Lambda}}_{\mathtt{B}})$ (dotted) intersect in a common point (gray) and their common orthogonal $\hat{\triangle}(\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}) \otimes \hat{\triangle}(\hat{\boldsymbol{\Lambda}}_{\mathtt{B}})$ (dashed, red) passes their intersection point parallel with the original lines. Its normalized polar $\hat{\boldsymbol{\Lambda}}_{\mathtt{AB}} \coloneqq \hat{\triangle}(\hat{\triangle}(\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}) \otimes \hat{\triangle}(\hat{\boldsymbol{\Lambda}}_{\mathtt{B}}))$ yields the sought common orthogonal of $\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}$ and $\hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$ (red). The numerical values of the example spears are $\hat{\boldsymbol{\Lambda}}_{\mathtt{A}} = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right) + \epsilon\left(\begin{smallmatrix}0\\-5/4\\0\end{smallmatrix}\right)$ with $\underline{\boldsymbol{a}}_{\mathtt{A}} = \left(\begin{smallmatrix}5/4\\0\\0\end{smallmatrix}\right)$ and $\hat{\boldsymbol{\Lambda}}_{\mathtt{B}} = \left(\begin{smallmatrix}0\\0\\-1\end{smallmatrix}\right) + \epsilon\left(\begin{smallmatrix}-5/4\\0\\0\end{smallmatrix}\right)$ with $\underline{\boldsymbol{a}}_{\mathtt{B}} = \left(\begin{smallmatrix}0\\5/4\\0\end{smallmatrix}\right)$; then $\hat{\triangle}(\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}) = \left(\begin{smallmatrix}0\\-1\\0\end{smallmatrix}\right) + \epsilon\left(\begin{smallmatrix}0\\0\\4/5\end{smallmatrix}\right)$ with $\underline{\boldsymbol{a}}_{\triangle\mathtt{A}} = \left(\begin{smallmatrix}0\\0\\4/5\end{smallmatrix}\right)$ and $\hat{\triangle}(\hat{\boldsymbol{\Lambda}}_{\mathtt{B}}) = \left(\begin{smallmatrix}-1\\0\\0\end{smallmatrix}\right) + \epsilon\left(\begin{smallmatrix}0\\0\\-4/5\end{smallmatrix}\right)$ with $\underline{\boldsymbol{a}}_{\triangle\mathtt{B}} = \left(\begin{smallmatrix}0\\0\\-4/5\end{smallmatrix}\right)$; and finally $\hat{\boldsymbol{\Lambda}}_{\mathtt{AB}} = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1\\-1\\0\end{smallmatrix}\right) + \frac{\epsilon}{\sqrt{2}}\left(\begin{smallmatrix}0\\0\\5/4\end{smallmatrix}\right)$ with $\underline{\boldsymbol{a}}_{\mathtt{AB}} = \left(\begin{smallmatrix}5/8\\5/8\\0\end{smallmatrix}\right)$. The signed distance is $\underline{s}_{\mathtt{AB}} = -\frac{5\sqrt{2}}{4}$ with (20).

With the canonic anchors (2), the foot points are computed by the robust formulae

$$
\begin{aligned}
\boldsymbol{c}_{\mathring{\mathtt{A}}\mathtt{B}} &= \pi^{\circ}_{\mathtt{A}}(\hat{\boldsymbol{\Lambda}}_{\mathtt{B}}) = \underline{\boldsymbol{a}}_{\mathtt{A}} + \frac{(\hat{\boldsymbol{n}}_{\mathtt{B}} \times \hat{\boldsymbol{n}}_{\mathtt{A}} \times \hat{\boldsymbol{n}}_{\mathtt{B}}) * \underline{\boldsymbol{d}}_{\mathtt{AB}}}{\|\hat{\boldsymbol{n}}_{\mathtt{A}} \times \hat{\boldsymbol{n}}_{\mathtt{B}}\|^2} \cdot \hat{\boldsymbol{n}}_{\mathtt{A}} \\
\boldsymbol{c}_{\mathtt{A}\mathring{\mathtt{B}}} &= \pi^{\circ}_{\mathtt{B}}(\hat{\boldsymbol{\Lambda}}_{\mathtt{A}}) = \underline{\boldsymbol{a}}_{\mathtt{B}} - \frac{(\hat{\boldsymbol{n}}_{\mathtt{A}} \times \hat{\boldsymbol{n}}_{\mathtt{B}} \times \hat{\boldsymbol{n}}_{\mathtt{A}}) * \underline{\boldsymbol{d}}_{\mathtt{AB}}}{\|\hat{\boldsymbol{n}}_{\mathtt{A}} \times \hat{\boldsymbol{n}}_{\mathtt{B}}\|^2} \cdot \hat{\boldsymbol{n}}_{\mathtt{B}}
\end{aligned}
\tag{24}
$$

For the exceptional case of coaligned lines $\hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \parallel \hat{\boldsymbol{\Lambda}}_{\mathtt{B}}$, the factors in the second summands of (24) are undefined, and $\underline{\boldsymbol{a}}_{\mathtt{A}}$ and $\underline{\boldsymbol{a}}_{\mathtt{B}}$ provide feasible results. If two lines intersect, their foot points equal and yield their intersection point $\boldsymbol{c}_{\mathring{\mathtt{A}}\mathring{\mathtt{B}}}$:

$$
\boldsymbol{c}_{\mathring{\mathtt{A}}\mathring{\mathtt{B}}} \coloneqq \boldsymbol{c}_{\mathring{\mathtt{A}}\mathtt{B}} = \boldsymbol{c}_{\mathtt{A}\mathring{\mathtt{B}}} = \hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \cap \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} \qquad \boldsymbol{c}_{\mathring{\mathtt{A}}\mathtt{B}} \neq \boldsymbol{c}_{\mathtt{A}\mathring{\mathtt{B}}} \Leftrightarrow \hat{\boldsymbol{\Lambda}}_{\mathtt{A}} \cap \hat{\boldsymbol{\Lambda}}_{\mathtt{B}} = \varnothing \tag{25}
$$

Outline. A toolset for analyzing the geometry (of pairs) of spears in general and exceptional cases is given with (6), (17), (18), (20), (21), (22), and (24).

4 Conclusions

Novel versions for fundamental geometric operations with oriented lines in space are worked out in the present paper. The refined computations build upon the idea of the canonic anchor – a line's unique anchor point with least distance to the origin – and the polarization operator. The novel formulae compute the normalized common orthogonal, the enclosed dual angle, foot points, and the intersection point for spear pairs in a robust manner. The implementations can be employed in particular for the analysis and control of mechanical systems operating in three-dimensional space. For the future, it is planned to pursue the analysis towards spear systems with more than two elements.

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