

Construction and Deformation of P-hedra Using Control Polylines

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Abstract In the 19th International Symposium on Advances in Robot Kinematics the author introduced a novel class of continuous flexible discrete surfaces and mentioned that these so-called P-hedra (or P-nets) allow direct access to their spatial shapes by three control polylines. In this paper we study this intuitive method, which makes these planar quad mesh mechanisms suitable for transformable design tasks by means of interactive tools. The construction of P-hedra from the control polylines can also be used for an efficient algorithmic computation of their isometric deformations. In addition we discuss flexion limits, bifurcation configurations, developable/flat-foldable patterns and tubular P-hedra.

1 Introduction

A planar quad-surface (PQ-surface) is a plate-and-hinge structure made of quadrilateral panels connected by rotational joints in the combinatorics of a square grid. Such a surface is called *continuous flexible* (or *rigid-foldable* or *isometric deformable*) if it can be continuously transformed by a change of the dihedral angles only. It is well known that the rigid-foldability of PQ-surfaces is not a property of the extrinsic geometry but of the intrinsic one [1], which is determined by the corner angles of the planar quads. Nonetheless, certain classes of rigid-foldable PQ-surfaces, namely V-hedra (and their related surfaces [2]) and T-hedra originally introduced by Sauer and Graf [3], allow for direct access to their spatial shape by control polylines.

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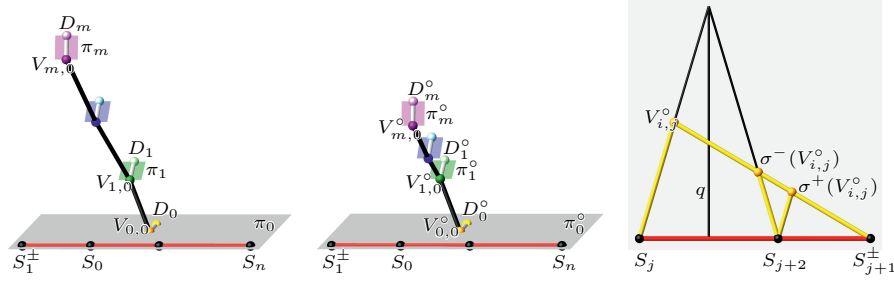


Fig. 1 (left) Input of a general P-hedron: trajectory polyline $V_{0,0}, V_{1,0}, \dots, V_{m,0}$, direction polyline $D_0, D_1, \dots, D_m$ and apex polyline $S_0, S_1^\pm, \dots, S_{n-1}^\pm, S_n$. (center) Input of the associated axial P-hedron: trajectory polyline $V_{0,0}^\circ, V_{1,0}^\circ, \dots, V_{m,0}^\circ$, direction polyline $D_0^\circ, D_1^\circ, \dots, D_m^\circ$ and apex polyline $S_0, S_1^\pm, \dots, S_{n-1}^\pm, S_n$. (right) Illustration of the two constructions related to the sign of $S_{j+1}^\pm$.

In Section 2 we show that this also holds for P-hedra which makes these planar quad mesh mechanisms [4] as suitable as V- and T-hedra (see [2, 5, 6] and the references therein) for transformable design tasks using interactive tools. In Section 3 we give an algorithm for their isometric deformations within the class of P-hedra and discuss possible bifurcation configurations and flexion limits, respectively (cf. Remarks 2 and 4). Moreover, in Section 4 we comment on developable/flat-foldable pattern, the construction of P-hedral tubes and future research.

2 Reconstruction of general P-hedra from three control polylines

In the following we describe how a general P-hedron can be reconstructed from three polylines illustrated in Fig. 1-left.

Given is a so-called trajectory polyline $V_{0,0}, V_{1,0}, \dots, V_{m,0}$. Moreover, we have a polyline formed by direction points $D_0, D_1, \dots, D_m$ with $V_{i,0} \neq D_i$. These points cannot be selected arbitrarily but the vector $\overrightarrow{V_{i,0}D_i}$ has to be orthogonal to a fixed direction, which can be assumed to be the z -direction of the fixed frame; i.e. $\langle D_i - V_{i,0}, z \rangle = 0$ where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product. As the lengths of the vectors $\overrightarrow{V_{i,0}D_i}$ are not of relevance, they can be assumed as unit-vectors.

The plane through $V_{i,0}$ and D_i which is parallel to the z -direction is called profile plane π_i . Moreover, we assume that no two consecutive profile planes

are identical¹ or parallel². Without loss of generality (w.l.o.g.) we can assume that the z -axis equals the intersection line of π_0 and π_1 . Moreover, we can assume w.l.o.g. that π_0 equals the xz -plane and that $V_{0,0}$ is located on the positive x -axis³. Beside the trajectory polyline and the direction polyline, we can select finite⁴ points $S_0, S_1, \dots, S_n$ on the z -axis; i.e. $S_i = (0, 0, z_i)^T$. In order to avoid degenerated cases we assume that three consecutive vertices S_j, S_{j+1} and S_{j+2} are always pairwise distinct. In addition, we assign to each of the points $S_1, \dots, S_{n-1}$ either a plus or a minus sign⁵, which results in the sequence $S_0, S_1^\pm, \dots, S_{n-1}^\pm, S_n$. We denote this polyline as apex polyline. This nomenclature becomes clear at the end of this section.

Finally, we assume that the trajectory polyline is not contained in the xy -plane because then the P-hedron belongs also to the class of T-hedra, which are already well studied and understood (cf. [3, 5, 7, 8]).

Using these three input polylines the related general P-hedron can be constructed in the following three steps:

1. We compute the input of the axial P-hedron associated with the general one as follows (cf. Fig. 1-center): We apply a translation τ_i to all points $V_{i,0}, \dots, V_{m,0}$ and points $D_i, \dots, D_m$ in direction $\vec{V}_{i-1,0}, \vec{V}_{i,0}$ in such a way that $\tau_i(\pi_i)$ contains the z -axis. By iterating this procedure for $i = 2, \dots, m$ we end up with the polylines $V_{0,0}^\circ = V_{0,0}, V_{1,0}^\circ = V_{1,0}, V_{2,0}^\circ, \dots, V_{m,0}^\circ$ and $D_0^\circ = D_0, D_1^\circ = D_1, D_2^\circ, \dots, D_m^\circ$ and the pencil of direction planes $\pi_0^\circ = \pi_0, \pi_1^\circ = \pi_1, \pi_2^\circ, \dots, \pi_m^\circ$. As we only applied translations, the following properties hold true:

$$V_{i,0}^\circ V_{i+1,0}^\circ \parallel V_{i,0} V_{i+1,0}, \quad V_{i,0}^\circ D_i^\circ \parallel V_{i,0} D_i, \quad \pi_i^\circ \parallel \pi_i. \quad (1)$$

This is the parallelism operation of Sauer and Graf [3, 7] mentioned in [9].

2. In each of the planes π_i° we proceed with an iterative composition of two possible linear constructions σ^+ and σ^- , respectively, to determine the point sequence $V_{i,0}^\circ, V_{i,1}^\circ, \dots, V_{i,n-1}^\circ$ (cf. Fig. 2-left). The linear mappings $\sigma^\pm$ are defined as follows (cf. Fig. 1-right) according to [9]:

- (+) There exists a central scaling σ^+ which maps S_j to S_{j+2} with center S_{j+1}^+ .
- (-) There exists a perspective collineation σ^- which maps S_j to S_{j+2} with center S_{j+1}^- and the bisector q of S_j and S_{j+2} as axis.

Then $V_{i,j+1}^\circ$ can be constructed from $V_{i,j}^\circ$ as $\sigma^\pm(V_{i,j}^\circ)$ for $j = 0, \dots, n-2$.

¹ $\pi_i = \pi_{i+1}$ and $V_{i,0} \neq V_{i+1,0}$ implies a bifurcation configuration (cf. Remark 4).

² $\pi_i \parallel \pi_{i+1}$ implies some special treatment as we get a translational surface-strip (cf. Section 4).

³ We assume that $V_{0,0}$ differs from the origin, in order to avoid a degenerated case.

⁴ As mentioned in [9] S_i can also be an ideal point but we do not consider this special case here.

⁵ The $\pm$ assignment can be done arbitrarily with exception of the case mentioned in Remark 1.

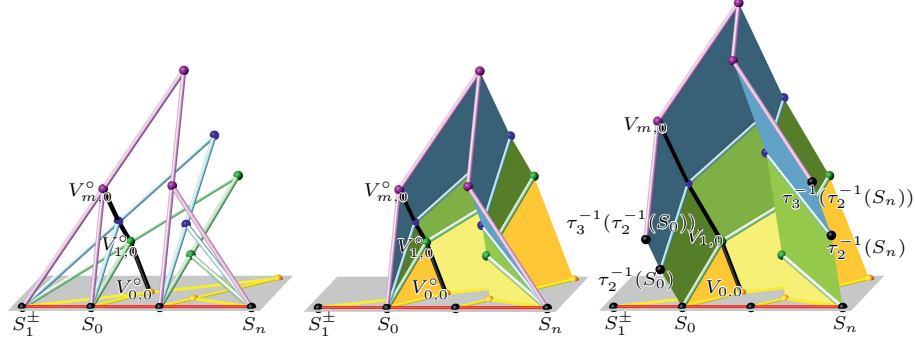


Fig. 2 (left) Iterative composition of the two possible linear constructions σ^+ and σ^- in π_i° with $V_{i,0}^\circ$ as starting point. Interpretation of the resulting point set as axial P-hedron (center) and the corresponding general P-hedron (right).

- By the end of the second step we already obtain all points of the axial P-hedron (cf. Fig. 2-center), where it can also be seen that the S_i are the apexes of cones.

For reconstruction of the general P-hedron we have to apply iteratively the translations τ_i^{-1} to all points $V_{i,0}^\circ, V_{i,1}^\circ, \dots, V_{i,n-1}^\circ, \dots, V_{m,0}^\circ, V_{m,1}^\circ, \dots, V_{m,n-1}^\circ$ for $i = 2, \dots, m$. In order to generate a boundary we also apply this series of transformations to the points S_0 and S_n , respectively (cf. Fig. 2-right).

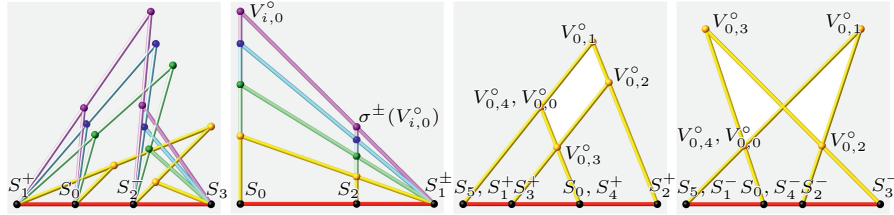


Fig. 3 (left) The point sets located in the plane π_i° for $i = 0, \dots, m$ can be interpreted as planar linkage L_i with mobility 1. If we rotate all planes π_i° for $i = 1, \dots, m$ into π_0° , we get the illustrated overconstrained planar linkage $\mathcal{L}$ discussed in [9]. (middle-left) Bifurcation configuration of Remark 2. Illustration of the linkage L_0 enclosing a parallelogram (middle-right) and an anti-parallelogram (right), respectively.

3 Isometric deformations within the class of P-hedra

As a result of the reconstruction done in Section 2, we can assume w.l.o.g. that the lengths of all edges of the general P-hedron and its associated axial

P-hedron are known. As the general P-hedron can be obtained from the axial one by step 3 of Section 2, we can restrict ourselves to the parametrization of the isometric deformation of the axial P-hedron, which is explained in the following two subsections.

3.1 Parametrizing the motion of the linkage L_0

We assumed that $V_{0,0}^\circ$ is located on the positive x -axis and this property should be kept during the deformation. We do not use this x -coordinate of $V_{0,0}^\circ$ as motion parameter t but the z -coordinate of S_0 or S_1 according to the following criterion:

$$\begin{array}{llll} \text{Case (a)} & t = z_0 & \text{for} & \|S_0 - V_{0,0}^\circ\| < \|S_1 - V_{0,0}^\circ\| \\ \text{Case (b)} & t = z_1 & \text{for} & \|S_0 - V_{0,0}^\circ\| > \|S_1 - V_{0,0}^\circ\| \end{array}$$

The reason for this choice is that under a continuous deformation of the configuration $S_0, V_{0,0}^\circ, S_1$ only the end point of the shorter bar can switch the sign of its z -coordinate. In order to avoid an unnecessary distinction⁶ of cases we assume that this z -coordinate equals the motion parameter t . Note that t_* indicates the time instant of the deformation which corresponds to the initial given configuration.

Until now neither case (a) nor case (b) is covering the possibility $\|S_0 - V_{0,0}^\circ\| = \|S_1 - V_{0,0}^\circ\|$. In this case we consider the smallest i with $\|S_0 - V_{i,0}^\circ\| \neq \|S_1 - V_{i,0}^\circ\|$. For $\|S_0 - V_{i,0}^\circ\| < \|S_1 - V_{i,0}^\circ\|$ we apply case (a); otherwise case (b). The equality $\|S_0 - V_{i,0}^\circ\| = \|S_1 - V_{i,0}^\circ\|$ cannot hold for all $i = 0, \dots, m$ due to the assumption formulated in the fourth paragraph of Section 2.

Remark 1 If $\|S_0 - V_{i,0}^\circ\| = \|S_1 - V_{i,0}^\circ\|$ holds for at least one $i \in \{0, \dots, m\}$ then the apexes polyline must be assigned with only pluses⁷; i.e. $S_0, S_1^+, \dots, S_{n-1}^+, S_n$, because otherwise some of the points $V_{i,1}^\circ, \dots, V_{i,n-1}^\circ$ would drop to infinity. This can easily be seen by adopting the mapping σ^- illustrated in Fig. 1-right to this special case. $\diamond$

After clarifying the choice of the motion parameter t we proceed as follows:

ad a) We start with $S_0(t)$ and then we parametrize $V_{0,0}^\circ(t) = (x_{0,0}(t), 0, 0)^T$ by $x_{0,0}(t) := \sqrt{\|V_{0,0}^\circ - S_0\|^2 - t^2}$. From this we can compute $S_1(t) = (0, 0, z_1(t))^T$ by $z_1(t) = \text{sgn}(z_1) \sqrt{\|V_{0,0}^\circ - S_1\|^2 - x_{0,0}(t)^2}$.

⁶ For the same reason we do not select the x -coordinate of $V_{0,0}^\circ$ as motion parameter.

⁷ Note that in this special case the linkage L_i (cf. Remark 3 and Fig. 3-left) is a scissor-like linkage.

ad b) In this case we start with $S_1(t)$ and parametrize $V_{0,0}^\circ(t) = (x_{0,0}(t), 0, 0)^T$ by $x_{0,0}(t) := \sqrt{\|V_{0,0}^\circ - S_1\|^2 - t^2}$. From this we can compute $S_0(t) = (0, 0, z_0(t))^T$ by $z_0(t) = \text{sgn}(z_0) \sqrt{\|V_{0,0}^\circ - S_0\|^2 - x_{0,0}(t)^2}$.

In both cases we end up with the three parametrized points $S_0(t)$, $V_{0,0}^\circ(t)$, $S_1(t)$. The remainder of this subsection is formulated in a way that it fits for both cases.

We proceed with the computation of $V_{0,1}^\circ(t) = (x_{0,1}(t), 0, z_{0,1}(t))^T$ by

$$V_{0,1}^\circ(t) = S_1(t) + \text{sgn}(k) \frac{\|V_{0,1}^\circ - S_1\|}{\|V_{0,0}^\circ(t) - S_1(t)\|} (V_{0,0}^\circ(t) - S_1(t)) \quad (2)$$

where $k := \langle V_{0,1}^\circ - S_1, V_{0,0}^\circ - S_1 \rangle$. Then we can compute $S_2(t)$ by

$$S_2(t) = V_{0,1}^\circ(t) + \text{sgn}(k) \frac{\|V_{0,1}^\circ - S_1\| \|V_{0,0}^\circ - S_0\|}{\|V_{0,0}^\circ - S_1\| \|S_0(t) - V_{0,0}^\circ(t)\|} M^\pm (S_0(t) - V_{0,0}^\circ(t)) \quad (3)$$

where the choice of $M^+ = \text{diag}(1, 1, 1)$ and $M^- = \text{diag}(1, 1, -1)$, respectively, depends on the sign $\pm$ associated with $S_1(t)$, which can be assumed w.l.o.g. to be the same as assigned to S_1 (cf. Remark 2).

Remark 2 The sign $\pm$ of $S_1^\pm(t)$ can only change under the isometric deformation in a configuration, where σ^+ and σ^- act in the same way on the points $V_{i,0}^\circ(t)$ for $i = 0, \dots, m$. Such a bifurcation can only happen in the configuration illustrated in Fig. 3-middle-left, but then the P-hedron also belongs to the class of T-hedra (cf. fourth paragraph of Section 2). Therefore a non-T-hedral P-hedron cannot have a bifurcation configuration implied by a change of the mappings σ^+ and σ^- . $\diamond$

An iteration of Eqs. (2) and (3) by rising the indices implies a parametrization of all points $S_0(t), \dots, S_n(t)$ and $V_{0,0}^\circ(t), \dots, V_{0,n-1}^\circ(t)$ located in the plane π_0° , which can be interpreted as an overconstrained planar linkage L_0 with mobility 1.

Remark 3 In the same way one can interpret the corresponding point sets in the planes π_i° as planar linkages L_i for $i = 1, \dots, m$. If we rotate all planes π_i° into π_0° , we get the overconstrained planar linkage $\mathcal{L}$ illustrated in Fig. 3-left. $\diamond$

3.2 Parametrizing the motion of the trajectory polyline

Let us start with the computation of $V_{1,0}^\circ(t) = (x_{1,0}(t), y_{1,0}(t), z_{1,0}(t))^T$. The three coordinates can be computed from the following system of equations:

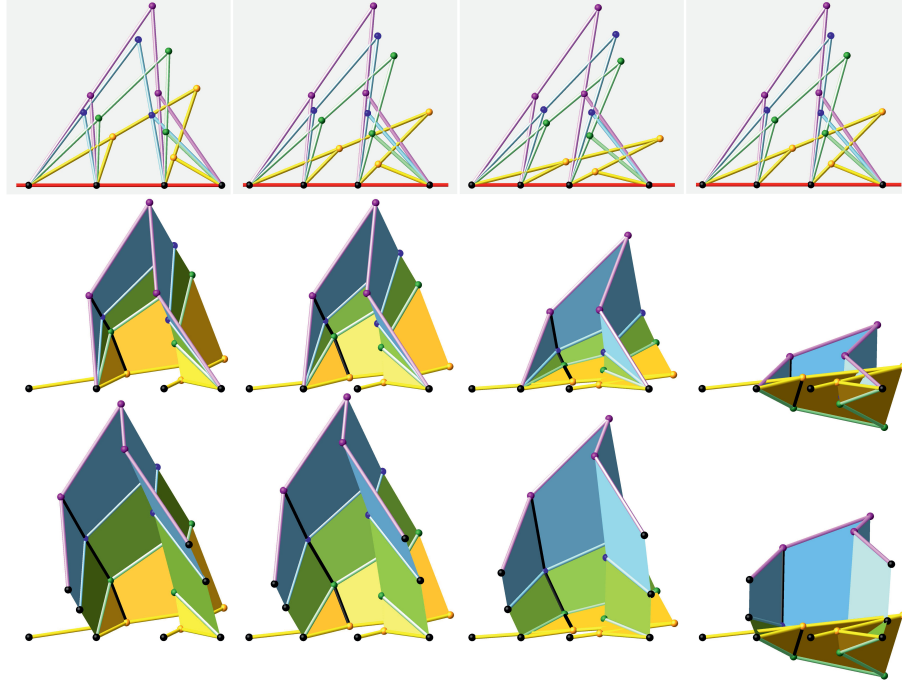


Fig. 4 Sequence of isometric deformation of the planar linkage $\mathcal{L}$ (cf. Remark 3) (top), axial P-hedron (middle) and general P-hedron (bottom). The corresponding animations can be downloaded from <https://www.geometrie.tuwien.ac.at/nawratil/publications.html>. The second column corresponds to the given configuration (parameter t_*) already illustrated in Figs. 2 and 3-left. The third column illustrates a flexion limit and the fourth column displays the other branch for the parameter t_* . Therefore the planar linkage $\mathcal{L}$ has the same configuration in the second and fourth column. Note that the P-hedra of the fourth column have self-intersections.

$$\begin{aligned} \|V_{1,0}^\circ(t) - S_0(t)\|^2 - \|V_{1,0}^\circ - S_0\|^2 &= 0, & \|V_{1,0}^\circ(t) - S_1(t)\|^2 - \|V_{1,0}^\circ - S_1\|^2 &= 0, \\ \|V_{1,0}^\circ(t) - V_{0,0}(t)\|^2 - \|V_{1,0}^\circ - V_{0,0}\|^2 &= 0, \end{aligned} \quad (4)$$

having two solutions, which are plane symmetric with respect to π_0° . We denote the branch containing $V_{1,0}^\circ$ by $V_{1,0}^\circ(t)$; i.e. $V_{1,0}^\circ(t_*) = V_{1,0}^\circ$; and the other one by $\underline{V}_{1,0}^\circ(t)$.

Now we select the branch $V_{1,0}^\circ(t)$. Using this parametrization we can compute $V_{2,0}^\circ(t)$ by solving an analogous system to Eqs. (4). We can iterate this procedure until we get the complete parametrized trajectory polyline $V_{0,0}^\circ(t), V_{1,0}^\circ(t), \dots, V_{m,0}^\circ(t)$.

Remark 4 Common configurations of the two branches $V_{i,0}^\circ(t)$ and $\underline{V}_{i,0}^\circ(t)$ are characterized by the coplanarity of the involved points (i.e. the profile

planes π_{i-1}° and π_i° coincide⁸); which corresponds to zeros of $\det(V_{i,0}^\circ(t) - S_0(t), V_{i,0}^\circ(t) - S_1(t), V_{i,0}^\circ(t) - V_{i-1,0}(t)) = 0$. If one computes all real zeros for $i = 1, \dots, m$ and sort them together with t_* we obtain the sequence: $t_\alpha \leq t_\beta \leq \dots \leq t_\lambda < t_* < t_\mu \leq \dots \leq t_\omega$. Note that $t_\alpha, \dots, t_\lambda$ and $t_\mu, \dots, t_\omega$ are candidates for the flexion limits $t_{\min}$ and $t_{\max}$, respectively, characterized by the fact that $t \in [t_{\min}, t_{\max}]$ implies real configurations and that an overshoot of these limits yield complex solutions of the corresponding system (4). Note that due to the assumption related to Footnote 1 the length of the flexion interval cannot be zero, i.e. there always exists a real flex out of the given configuration. We can follow this isometric deformation until the flexion limits are reached. As they are also bifurcation configurations we can switch over to the other branch and flex back (cf. Fig. 4). $\diamond$

The parametrization of the remaining vertices of the axial P-hedron can be completed by using $V_{i,0}^\circ(t)$ as starting point for the iteration of the analogous equations to (2). In this way we get the points $V_{i,1}^\circ(t), \dots, V_{i,n-1}^\circ(t)$ for $i = 1, \dots, m$.

A test implementation of the given algorithm was done for verification in Maple, which was also used to produce the example illustrated in Fig. 4. For real-time interactive handling it is planned to implement this algorithm⁹ in a Rhino/Grasshopper plugin. Such a plugin can also be used to approximate the isometric deformation of semi-discrete P-hedra, by discretizing a given smooth input trajectory into a polyline with sufficiently small line-segments.

4 Final remarks

- **Developable pattern** play an important role in origami and fabrication. Developable (but also flat-foldable) pattern can be considered as special P-hedra where all bifurcation possibilities arise at the same time; i.e. all π_i collapse into one plane. Clearly one can construct the P-hedron in this developed (flat-folded) configuration for which $V_{i,j} = V_{i,j}^\circ$ and $\pi_i = \pi_i^\circ$ hold true. But now one cannot be sure that a real flex out of the constructed configuration exists, as the relation $t_{\min} = t_* = t_{\max}$ (in terms of Remark 4) could hold true. For the existence of a real flex we have to show that the linkage $\mathcal{L}$ (cf. Remark 3) can be associated with an infinitesimal motion ($\neq$ instantaneous standstill) in a way that the distances between $V_{i,j}^\circ$ and $V_{i+1,j}^\circ$ for all $i = 0, \dots, m-1$ and $j = 0, \dots, n-1$ do not expand instantaneously, as otherwise the PQ-mesh would tear apart. This non-expansion can easily be checked by the criterion:

⁸ This characterizes also the flexion limits/bifurcation configurations of T-hedra (cf. [3, 5, 7]).

⁹ The special case mentioned in Footnote 4 has to be coded separately.

$$\langle v(V_{i,j}^\circ), V_{i,j}^\circ - V_{i+1,j}^\circ \rangle + \langle v(V_{i+1,j}^\circ), V_{i+1,j}^\circ - V_{i,j}^\circ \rangle \leq 0 \quad (5)$$

where $v(\cdot)$ denotes the velocity of the point associated with the planar linkage $\mathcal{L}$. The determination of these velocities is a standard procedure in the kinematics of planar mechanisms, which can even be done in a pure graphical way (e.g. [10]).

Remark 5 Developable/flat-foldable P-hedra belong to the conic equimodular class according to [1]. Their semi-discrete analogues belong to nets discussed in [11]. $\diamond$

• **P-hedral tubes** are further interesting objects, as they can be regarded as building blocks of rigid-foldable meta-materials/surfaces (cf. [6]). They are obtained by constructing a linkage L_0 in a way that $V_{0,0}^\circ(t) = V_{0,n-1}^\circ(t)$ holds true for all $t \in [t_\lambda, t_\mu]$. For $n = 5$ this can only be the case¹⁰ for a parallelogram or an anti-parallelogram, where the symmetry line of the latter one has to be orthogonal to the axis of the axial P-hedron (see Fig. 3). The parallelogram results in a rigid-foldable prismatic tube and the anti-parallelogram in a composition of plane-symmetric Bricard octahedra.

In analogy to [6] we can also combine these flexible tubes by edge-sharing and/or the aligned-coupling of faces. By deleting the common faces of the latter coupling one can produce more complicated P-hedral tubes than the two mentioned above.

• **Future research** is dedicated to the study of (i) zipper couplings of P-hedral tubes (cf. [6]) and (ii) isometric deformations of translational surfaces contained in the class of general P-hedra, which were mentioned in Footnote 2.

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¹⁰ Both cases also appear for T-hedra, where in addition the deltoidal case exists (cf. [6]).

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