On the Displacement Variety of the RSR Linkage

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Abstract The RSR limb has been used as a component of several robotics mechanisms. In this work we show that the space of possible rigid displacements that the end-effector of such a linkage can achieve is given by the intersection of the Study quadric with a quartic hypersurface in $\mathbb{P}^7$. The computation involves casting the problem as a geometric constraint problem. We seek the rigid displacements that move a circle so that it remains in contact with a fixed circle. The computation is facilitated by a classical representation of circles in space by lines in 4-dimensions.

1 Introduction

Consider a serial linkage consisting of an open chain of an R, an S and a final R joint. The displacement variety of such a limb is the variety of all rigid-body displacements that the end-effector of the linkage can adopt relative to its fixed home configuration. Since the linkage has 5 degrees of freedom we may expect that the displacement variety is the intersection of the Study quadric with another hypersurface in $\mathbb{P}^7$.

This limb has been considered in recent studies as a component in various parallel manipulators, in particular, the Canfield joint [1]. In [2] several kinematic equivalent linkages were synthesised to avoid the use of a spherical joint and in [9] it was observed that the chain was equivalent to a certain origami construction and used to design a continuum robot.

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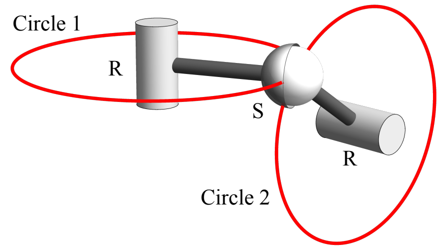

Fig. 1 Two Circles determined by an RSR Chain.

To turn this in to a geometrical constraint problem fix a circle with centre lying on the axis of the first R-joint and passing through the centre of the S-joint in its home position. For the second R-joint we have another circle defined in the same way. As the chain moves the first circle stays fixed but the second circle moves with the second R-joint. The two circles must, however, remain in contact at the centre of the S-joint. See figure 1.

Classically, spheres can be represented using pentaspherical coordinates,

$$\mathring{c} = \begin{pmatrix} c_x \\ c_y \\ c_z \\ -1/2 \\ (\rho^2 - |\vec{c}|^2)/2 \end{pmatrix} = \begin{pmatrix} \vec{c} \\ -1/2 \\ (\rho^2 - |\vec{c}|^2)/2 \end{pmatrix},$$

where $\vec{c} = (c_x,\ c_y,\ c_z)^T$ are the coordinates of the centre of the sphere and ρ its radius, see [3, 8]. Multiplying these coordinates by a non-zero constant gives the same sphere, these are coordinates in a 4-dimensional projective space, $\mathbb{P}^4$.

In this formalism a point $\vec{r} = (x,\ y,\ z)^T$ can be given by a vector

$$\hat{r} = \begin{pmatrix} x \\ y \\ z \\ |\vec{r}|^2 \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{r} \\ |\vec{r}|^2 \\ 1 \end{pmatrix}.$$

This is so that the condition for the point to lie on the sphere is, $\hat{r}^T\mathring{c} = 0$, since the equation of a sphere with centre $\vec{c}$ and radius ρ is given by

$$(\vec{r} - \vec{c})^2 = \rho^2.$$

Now, circles in space can be represented by pencils of spheres. The set of spheres that a given circle lie on comprise a linear system of spheres. Hence, we may represent circles by lines in $\mathbb{P}^4$, see [5].

2 Plücker Coordinates of Circles

The set of lines in a $\mathbb{P}^4$ is a variety known as a Grassmannian and denoted $G(1, 4)$. According to Harris [4], this is a six dimensional variety. The Plücker embedding maps it to a degree 5 subvariety of $\mathbb{P}^9$.

Suppose that $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ and $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ are two points in $\mathbb{P}^4$ then the Plücker coordinates of the line through these two points are,

$$p_{ij} = \alpha_i \beta_j - \alpha_j \beta_i, \qquad i, j = 1, 2, \dots, 5, \quad i \neq j. \tag{1}$$

There are 20 such Plücker coordinates but only 10 linearly independent ones since $p_{ij} = -p_{ji}$. Not all choices of Plücker coordinates give lines in $\mathbb{P}^4$, the Grassmannian is defined as a projective algebraic variety by the Plücker relations. In this case these are given by five quadratic equations,

$$\begin{aligned}
p_{23}p_{45} - p_{24}p_{35} + p_{25}p_{34} &= 0,\\
p_{31}p_{45} + p_{14}p_{35} - p_{15}p_{34} &= 0,\\
p_{12}p_{45} - p_{14}p_{25} + p_{15}p_{24} &= 0,\\
p_{12}p_{35} + p_{31}p_{25} + p_{15}p_{23} &= 0\\
p_{12}p_{34} + p_{31}p_{24} + p_{14}p_{23} &= 0.
\end{aligned} \tag{2}$$

The ten linearly independent Plücker coordinates can be grouped into 3 sets of 3 coordinates plus one extra.

For a circle determined by a pair of spheres $\mathring{c}_1 = \left(\vec{c}_1^{\,T}, -1/2, (\rho_1^2 - |\vec{c}_1|^2)/2\right)^T$ and $\mathring{c}_2 = \left(\vec{c}_2^{\,T}, -1/2, (\rho_2^2 - |\vec{c}_2|^2)/2\right)^T$ we get,

$$\begin{pmatrix} p_{14} \\ p_{24} \\ p_{34} \end{pmatrix} = \frac{-1}{2}(\vec{c}_1 - \vec{c}_2) = \vec{u} \qquad \text{and} \qquad \begin{pmatrix} p_{23} \\ p_{31} \\ p_{12} \end{pmatrix} = \vec{c}_1 \times \vec{c}_2 = \vec{v}.$$

where $\vec{u}$ is normal to the plane that the circle lies in and $\vec{v}$ is the moment of the line joining the centres of the spheres. The final group of three components is,

$$\begin{pmatrix} p_{15} \\ p_{25} \\ p_{35} \end{pmatrix} = \frac{1}{2}\left(\vec{c}_1(\rho_2^2 - |\vec{c}_2|^2) - \vec{c}_2(\rho_1^2 - |\vec{c}_1|^2)\right) = \vec{w}.$$

The tenth component is given by,

$$p_{45} = \frac{-1}{4}\left((\rho_2^2 - |\vec{c}_2|^2) - (\rho_1^2 - |\vec{c}_1|^2)\right) = \delta.$$

This quantity is related to the perpendicular distance between the origin and the plane determined by the circle.

The Plücker coordinates of a circle in this order will be written as a 10-component vector partitioned as

$$\overline{p} = \begin{pmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \\ \delta \end{pmatrix}.$$

The Plücker relations (2) can be written in terms of these vectors. The first three of the relations can be combined into a single vector equation,

$$\vec{u} \times \vec{w} - \delta \vec{v} = \vec{0}.$$

The final pair of relations can be written as scalar equations,

$$\vec{u} \cdot \vec{v} = 0 \qquad \text{and} \qquad \vec{v} \cdot \vec{w} = 0.$$

3 The Condition for Two Circles to Meet

Assume that each of the two circles is given as the intersection of a pair of spheres. So, there are in all four spheres which we may assume are given as,

$$\mathring{\mathbf{c}}_1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix}, \quad \mathring{\mathbf{c}}_2 = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix}, \quad \mathring{\mathbf{c}}_3 = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \end{pmatrix}, \quad \mathring{\mathbf{c}}_4 = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{pmatrix}.$$

If there is a point $\hat{r} = (\sigma_1,\, \sigma_2,\, \sigma_3,\, \sigma_4,\, \sigma_5)^T$ that lies on the two circles it will lie on all four spheres and so will satisfy the four linear equations,

$$\hat{r}^T \mathring{\mathbf{c}}_1 = 0, \quad \hat{r}^T \mathring{\mathbf{c}}_2 = 0, \quad \hat{r}^T \mathring{\mathbf{c}}_3 = 0, \quad \hat{r}^T \mathring{\mathbf{c}}_4 = 0.$$

These can be solved by determinants and written in terms of the Plücker coordinates of the circles. Suppose the Plücker coordinates of the first circle, the intersection of the first two spheres are $\vec{u}_1$, $\vec{v}_1$, $\vec{w}_1$ and δ_1 and similar for the second circle — the intersection of the other two spheres. The solution for the linear equations is given by the polarisation of the Plücker relations, (2),

$$\begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = -\lambda(\vec{u}_1 \times \vec{w}_2 + \vec{u}_2 \times \vec{w}_1 - \delta_1 \vec{v}_2 - \delta_2 \vec{v}_1),$$

$$r_4 = -\lambda(\vec{v}_1 \cdot \vec{w}_2 + \vec{v}_2 \cdot \vec{w}_1) \quad \text{and} \quad r_5 = \lambda(\vec{u}_1 \cdot \vec{v}_2 + \vec{u}_2 \cdot \vec{v}_1).$$

Here λ is an arbitrary non-zero constant and hence irrelevant for the homogeneous coordinates of the point. These coordinates may not be the coordinates of a point in space. For this point in $\mathbb{P}^4$ to represent a point in space we must have that $r_x^2 + r_y^2 + r_z^2 - r_4 r_5 = 0$. Hence, the condition for a pair of circles to meet at a point in space is

$$|\vec{u}_1 \times \vec{w}_2 + \vec{u}_2 \times \vec{w}_1 - \delta_1 \vec{v}_2 - \delta_2 \vec{v}_1|^2 + (\vec{u}_1 \cdot \vec{v}_2 + \vec{u}_2 \cdot \vec{v}_1)(\vec{v}_1 \cdot \vec{w}_2 + \vec{v}_2 \cdot \vec{w}_1) = 0. \quad (3)$$

Notice, this equation is separately of degree 2 in the Plücker coordinates of each circle. The equation can be written in a vector-matrix form as

$$\overline{p}_2^T \Big(Q_a + Q_b \Big) \overline{p}_2 = 0, \quad (4)$$

where $\overline{p}_2^T$ are the Plücker coordinates for the second circle, Q_a and Q_b contain only Plücker coordinates for the first circle. In partitioned form the two 10×10 symmetric matrices can be written,

$$Q_a = \begin{pmatrix} -W_1^2 & -\delta_1 W_1 & -W_1 U_1 & -W_1 \vec{v}_1 \\ \delta_1 W_1 & \delta_1^2 I & \delta_1 U_1 & \delta_1 \vec{v}_1 \\ -U_1 W_1 & -\delta_1 U_1 & -U_1^2 & -U_1 \vec{v}_1 \\ \vec{v}_1^T W_1 & \delta_1 \vec{v}_1^T & \vec{v}_1^T U_1 & |\vec{v}_1|^2 \end{pmatrix} \text{ and } Q_b = \frac{1}{2} \begin{pmatrix} 0 & \vec{v}_1 \vec{w}_1^T & \vec{v}_1 \vec{v}_1^T & 0 \\ \vec{w}_1 \vec{v}_1^T & \vec{u}_1 \vec{w}_1^T + \vec{w}_1 \vec{u}_1^T & \vec{u}_1 \vec{v}_1^T & 0 \\ \vec{v}_1 \vec{v}_1^T & \vec{v}_1 \vec{u}_1^T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where I is the 3×3 identity matrix, and U_1 is the 3×3 anti-symmetric matrix corresponding to the vector $\vec{u}_1$ and so forth.

4 The Displacement Variety

Under a rigid-body displacement points in space transform according to, $\vec{c} \longmapsto R\vec{c}+\vec{t}$, where R is the usual 3×3 rotation matrix and $\vec{t}$ is the translation vector of the displacement. The normal to the plane that a circle lies on, is a free vector so transforms as $\vec{u} \longmapsto R\vec{u}$. The moment of the circle's axis transforms as $\vec{v} \longmapsto 2TR\vec{u}+R\vec{v}$, where T it the 3×3 anti-symmetric matrix corresponding to the translation $\vec{t}$. The transformation of the remaining Plücker coordinates of a circle can also be found and these relations can be combined into a single relation, $\overline{p} \longmapsto G\overline{p}$, where G is the 10×10 matrix partitioned as

$$G = \begin{pmatrix} R & 0 & 0 & 0 \\ 2TR & R & 0 & 0 \\ (|\vec{t}|^2 I - 2\vec{t}\vec{t}^T)R & -TR & R & -2\vec{t} \\ \vec{t}^T R & 0 & 0 & 1 \end{pmatrix}. \quad (5)$$

Now, suppose that $\overline{p}_1$ and $\overline{p}_2$ are the Plücker coordinates of a pair of circles determined by the home configuration of an RSR chain. The first circle will be fixed but the second circle can perform rigid-body displacements but restricted to those that keep the pair of circles in contact, equation (4). The set of group elements satisfying this constraint will therefore satisfy the equation

$$\overline{p}_2^T G^T \Big(Q_a + Q_b \Big) G \overline{p}_2 = 0. \tag{6}$$

This is an equation in terms of the components of the 3×3 rotation matrix R, and translation vector $\vec{t}$. It is usually more useful to have an equation in terms of the Study parameters, $a_0, \dots, a_3,\ c_0, \dots, c_3$, (see [6] for details of dual quaternions and the Study parameters). This can be done by substituting for R and $\vec{t}$ using the relations given in, among many other places, [7]. For example,

$$R = \frac{1}{\Delta}\begin{pmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & 2(a_1a_2 - a_0a_3) & 2(a_1a_3 + a_0a_2) \\ 2(a_1a_2 + a_0a_3) & a_0^2 - a_1^2 + a_2^2 - a_3^2 & 2(a_2a_3 - a_0a_1) \\ 2(a_1a_3 - a_0a_2) & 2(a_2a_3 + a_0a_1) & a_0^2 - a_1^2 - a_2^2 + a_3^2 \end{pmatrix},$$

where $\Delta = a_0^2 + a_1^2 + a_2^2 + a_3^2$ and

$$\vec{t} = \frac{2}{\Delta}\begin{pmatrix} a_0c_1 - a_1c_0 + a_2c_3 - a_3c_2 \\ a_0c_2 - a_1c_3 - a_2c_0 + a_3c_1 \\ a_0c_3 + a_1c_2 - a_2c_1 - a_3c_0 \end{pmatrix}.$$

The other elements of G can also be computed. In doing so the quadratic $Q_S = a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3$ can be set to zero. This is justified as we are only interested in rigid-displacements and these lie on the Study quadric, $Q_S = 0$. The results all have the common factor $(1/\Delta)$, since the Study parameters are homogeneous coordinates in a $\mathbb{P}^7$, we can multiply the matrix G in (5) by Δ to produce a matrix all of whose elements are quadratic in the Study parameters. From this it is clear that substituting these results into (6) will produce a homogeneous equation of degree 4 in the Study parameters. Together with the Study quadric, this defines a projective algebraic variety containing all the rigid displacements that the end-effector of an RSR chain can adopt. However, the general form of the polynomial has a large number of terms, so the full, general polynomial will not be given here. Rather, we look at a simple example, which illustrates the idea.

5 Example

Consider the home configuration of an RSR linkage illustrated in figure 2. We need to find the Plücker coordinates of the two circles determined by this configuration of the linkage. First notice that, the S-joint is located at the origin, so the planes containing each circles also contain the origin. Thus both δ_1 and δ_2 are zero. Next, the direction of the axes of both circles lie in the z-direction. So,

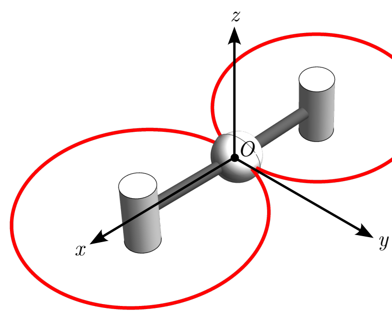

Fig. 2 Home Configuration of the Example RSR Linkage.

$$\vec{u}_1 = \vec{u}_2 = \begin{pmatrix} 0 \\ 0 \\ -1/2 \end{pmatrix}.$$

The moments of these axes are then

$$\vec{v}_1 = \begin{pmatrix} 0 \\ d_1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ -d_2 \\ 0 \end{pmatrix}.$$

Finally, notice that the position vector of the centre of any sphere which passes through the origin must have squared length equal to the square of the sphere's radius. Hence, both $(\rho_1^2 - |\vec{c}_1|^2) = 0$ and $(\rho_2^2 - |\vec{c}_2|^2) = 0$. Hence, both $\vec{w}_1 = \vec{0}$ and $\vec{w}_2 = \vec{0}$, and $\delta_1 = \delta_2 = 0$. Thus the two circles have Plücker coordinates

$$\overline{p}_1 = \begin{pmatrix} -1/2\vec{k} \\ d_1\vec{j} \\ \vec{0} \\ 0 \end{pmatrix} \quad \text{and} \quad \overline{p}_2 = \begin{pmatrix} -1/2\vec{k} \\ -d_2\vec{j} \\ \vec{0} \\ 0 \end{pmatrix}.$$

This enables us to compute the 10×10 symmetric matrix Q. We can also substitute the entries of G for the Study parameters using the results of the previous section. Finally, substituting all these results into equation (6) then gives the equation for the displacement variety,

$$\begin{aligned}\Big(\big(2c_0c_2 + c_0a_3(d_1 - d_2)\big) + (a_1c_2 - a_0c_3)(d_1 + d_2) + \big(2c_1c_3 - a_2c_1(d_1 - d_2)\big)\Big)^2 \\ +4d_1\big(c_0c_1 - c_2c_3 + (a_1c_1 + a_3c_3)d_2\big)\Big(\big(2a_2c_2 + a_2a_3(d_1 - d_2)\big) + \big(2a_1c_1 - a_0a_1(d_1 + d_2)\big)\Big) \\ + 4\big(c_0c_1 - c_2c_3 + (a_1c_1 + a_3c_3)d_2\big)^2 = 0.\end{aligned}$$

This equation clearly has degree 4 in the Study parameters. Moreover, we can see that setting $c_0 = c_1 = c_2 = c_3 = 0$ reduces the equation to $0 = 0$. This means that the A-plane of pure rotations about the origin lies in the displacement variety. This is consistent with the fact that the centre of the S-joint was located at the origin in the home position, so any rotation about the origin can be performed by the end-effector of the chain.

6 Conclusions

The intersection of this displacement variety with the A-plane of unphysical displacements is of interest. This A-plane is given by $a_0 = a_1 = a_2 = a_3 = 0$, substituting this into the equation above gives the equation of the intersection as $(c_0^2+c_3^2)(c_1^2+c_2^2) = 0$. So, the intersection is given by four 2-planes, $c_3 = \pm ic_0$, $c_2 = \pm ic_1$, where i is the imaginary unit here. The four planes meet in 6 lines which comprise the edges of a tetrahedron. Four of these lines, $(c_3 = ic_0,\ c_2 = ic_1)$, $(c_3 = ic_0,\ c_2 = -ic_1)$, $(c_3 = -ic_0,\ c_2 = ic_1)$ and $(c_3 = -ic_0,\ c_2 = -ic_1)$ lie in the complex 2-sphere $c_0^2 + c_1^2 + c_2^2 + c_3^2 = 0$, hence so do the vertices of the tetrahedron. The importance of these features, the infinite A-plane and the complex 2-sphere contained in it, are that they are invariants under the action of the group of rigid-body displacements $SE(3)$ on the Study quadric. Hence we can expect such an arrangement of planes and lines to be a feature of the displacement variety of any RSR limb.

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