Total Orientation and Total Pose in the Plane

Gregory S. Chirikjian

Abstract In robot kinematics, the use of $SO(n)$ and $SE(n)$ respectively to describe rotations and rigid-body displacements in n-dimensional Euclidean space is now the standard dogma. But in some scenarios it is useful to count the number of revolutions that a robot undergoes in addition to the terminal state that it reaches. This leads to the concept of total orientation and total pose described here. In the planar case, these quantities are the universal covering groups of the Lie groups $SO(2)$ and $SE(2)$. It is shown how these universal covers can not only add value, but are actually simpler objects to deal with than their more familiar counterparts. A new algebraic structure which generalizes better to the three-dimensional case is also discussed

1 Introduction

Theoretical kinematics and its applications in robotics are well established, with $SE(n)$ and its subgroups being the primary objects of study [1]-[10]. Other than pure translations, rotations in the plane are perhaps the simplest kinematic object. The underlying premise in the way that rotations are described is that a rotation by 0 and by 2π radians represents exactly the same quantity. In one sense that is true, but in another it is not. The distinction is reminiscent of a phrase from the Pink Floyd song 'Time': "..., The sun is the same in a relative way, but you're older ...". Yes, as the Earth makes one full rotation 3PM today is equivalent to 3PM tomorrow in some ways, but not all.

At a technical level, given rotational paths defined by rotation angle $\theta_1(t) = 2\pi t$ and $\theta_2(t) = 0$, the standard paradigm says that the final ori-

G. S. Chirikjian
Mohamed bin Zayed University of Artificial Intelligence, Abu Dhabi, Abu Dhabi, AE
e-mail: gregory.chirikjian@mbzuai.ac.ae

entation at $t = 1$ or $t = 10$ is the same in both cases. Yet we know that for some robots that have power cords running along their trunk that to execute multiple full rotations by 2π is actually not the same.

This distinction *could* be made by keeping *paths* in the Lie groups $SO(2)$ and $SE(2)$, but the space of paths is an infinite-dimensional object (which is also associative under the operation of concatenation of paths).

Throughout this presentation the case is made that there is nothing lost by keeping $\theta \in \mathbb{R}$, the universal cover of $SO(2)$. And regular arithmetic can be used to add total angles rather than arithmetic modulo 2π

And likewise, rather than representing planar motions as elements of

$$SE(2) \,=\, \mathbb{R}^2 \rtimes SO(2)$$

we can consider them as elements of the *universal cover*[1] of $SE(2)$,

$$\widetilde{SE(2)} \,=\, \mathbb{R}^2 \rtimes \mathbb{R}\,.$$

The only difference in the group laws is that total angles add, rather than computing modulo 2π, but the big difference is that the underlying set is now identifyiable with three-dimensional Euclidean space.

These are actually simpler objects to deal with than $SO(2)$ and $SE(2)$ themselves, in addition to being more descriptive. For example, if one wants to describe orientational uncertainty, much time and energy has been spent on trying to decide what to do with the tails of a Gaussian distribution (e.g., clip them and renormalize, or wrap them around the circle, or abandon altogether in favor of using Fisher or Bingham distributions, etc.) and how to define appropriate concepts of mean and covariance.

If one wishes to describe orientational uncertainty in the planar case one can do so in terms of total orientation with a standard Gaussian distribution on the universal cover as

$$\tilde{\rho}(x \,|\, \tilde{\mu}, \tilde{\sigma}^2) \,=\, \frac{1}{\sqrt{2\pi}\tilde{\sigma}} e^{-(x-\tilde{\mu})^2/2\tilde{\sigma}^2} \tag{1}$$

with $x \in \mathbb{R}$, or one can describe orientation on the circle by wrapping this around the circle as

$$\rho(\theta \,|\, \tilde{\mu}, \tilde{\sigma}^2) \,\doteq\, \sum_{k\in\mathbb{Z}} \tilde{\rho}(\theta + 2\pi k \,|\, \tilde{\mu}, \tilde{\sigma}^2) \,=\, \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-\tilde{\sigma}^2 n^2/2} \cos(n(\theta - \tilde{\mu})) \tag{2}$$

to get a periodized version. That is, nothing is lost by working in the universal cover, and it is backward compatible. If one has any probability density on

[1] This is a concept from the mathematical field of Algebraic Topology [11, 12]. In Theoretical Physics, related concepts from this field such as winding number, braid groups, and homotopy classes of paths are used to describe how certain particles move relative to each other, such as in [13, 14], which is a different goal than that here.

$\mathbb{R} \cong \widetilde{SO(2)}$, just wrap it around like in the case of the Gaussian above. Moreover, propagating uncertainty by convolution has the nice property that

$$\sum_{k \in \mathbb{Z}} (\tilde{\rho}_1 * \tilde{\rho}_2)(\theta + 2\pi k) = (\rho_1 \circledast \rho_2)(\theta)$$

where $*$ and $\circledast$ are respectively convolution on $\mathbb{R}$ and S^1. That is, the result of periodized convolution on the line is the same as circular convolution of two periodized distributions, which is true in general, not only for Gaussians. On the other hand, if one limits oneself to $SO(2)$ from the beginning, going the other direction is more challenging.

2 Mathematical Details for Planar Rotations

As is well known, rotations in the plane can be described as 2×2 rotation matrices, $R(\theta)$, or equivalently as the complex number $e^{i\theta}$. Either way, an exponential map takes each $\theta \in \mathbb{R}$ and bins it into a coset $\theta + \mathbb{Z} \in \mathbb{R}/2\pi\mathbb{Z}$.

This identifies the compact Lie group $SO(2)$ with the unit circle in the complex plane

$$S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\},$$

and in particular every element of $SO(2)$ has a unique representative $e^{i\theta}$ with $\theta \in [0, 2\pi)$. That is,

$$F_{\mathbb{R}/2\pi\mathbb{Z}} \doteq [0, 2\pi) \subset \mathbb{R}$$

is a fundamental domain within the original

The universal covering group of $SO(2)$ is the additive group $(\mathbb{R}, +)$ with covering map

$$\pi : \mathbb{R} \longrightarrow SO(2), \qquad \pi(x) = e^{ix}.$$

The kernel of this covering map is the discrete central subgroup

$$\ker \pi = 2\pi\mathbb{Z} = \{2\pi k : k \in \mathbb{Z}\},$$

and we have the standard identification

$$SO(2) \cong \mathbb{R}/2\pi\mathbb{Z}.$$

For most Lie groups, the exponential is well behaved, but the logarithm is problematic. For example the logarithm of a rotation $R(\pi)$ is ill defined, and more generally $\log(\exp X) \neq X$. But for $\widetilde{SO(2)}$ it is the case that $\log(\exp X) = X$. And more than that, it can be identified with its own Lie algebra: $\widetilde{SO(2)} \cong so(2)$.

The story of $SO(2)$ could end here, but if so there would be no connection or hint as to what to do about total orientation in 3D. The next section presents an alternative which has potential to generalize better to higher dimensions.

3 An Alternative Group Structure for Total Planar Orientation

While $(\mathbb{R}, +)$ is the simplest and most natural object to use for total orientation in the plane, it does not generalize well to higher dimensions. The universal cover of $SO(3)$ is $SU(2)$ (or, equivalently, the unit quaternions), and not $(\mathbb{R}^3, +)$. But what we can do in the 3D case is to measure the winding number and consider total orientations in higher dimensions as pairs of the form (q, k) where $q \in SU(2)$ and $k \in \mathbb{Z}$. The exposition below shows how complicated this becomes even in the planar case, but provides a path forward for considering higher dimensions.

3.1 The Operation $\oplus$ on $[0, 2\pi)$

It is convenient to introduce a binary operation $\oplus$ on the interval $[0, 2\pi)$ that encodes addition of angles modulo 2π together with a canonical choice of representative in $[0, 2\pi)$. This is just the usual circular (mod 2π) arithmetic.

Definition 1 For $\theta, \phi \in [0, 2\pi)$ define

$$N(\theta, \phi) := \left\lfloor \frac{\theta + \phi}{2\pi} \right\rfloor \in \{0, 1\},$$

(where $\lfloor \cdot \rfloor$ is the standard floor function) and

$$\theta \oplus \phi := \theta + \phi - 2\pi\, N(\theta, \phi) \in [0, 2\pi).$$

By construction, $\theta \oplus \phi$ is the unique representative in $[0, 2\pi)$ of the angle $(\theta + \phi)$ mod 2π.

It is well known that $([0, 2\pi), \oplus)$ is a commutative group isomorphic to $SO(2)$.

3.2 The Transported Group Law on the Set $SO(2) \times \mathbb{Z}$

We now define a nontrivial group law on the Cartesian set $SO(2) \times \mathbb{Z}$ by transporting addition on $\mathbb{R}$ through the bijection

$$\Phi : SO(2) \times \mathbb{Z} \longrightarrow \mathbb{R}\,.$$

Definition 2 For $\theta, \phi \in [0, 2\pi)$ and $k, \ell \in \mathbb{Z}$ define

$$(\theta, k) \circ (\phi, \ell) := (\theta \oplus \phi,\ k + \ell + N(\theta, \phi))\,,$$

where $N(\theta, \phi)$ and $\theta \oplus \phi$ are as above.

Equivalently, this is the unique operation such that

$$\Phi\big((\theta, k) \circ (\phi, \ell)\big) = \Phi(\theta, k) + \Phi(\phi, \ell)$$

for all $(\theta, k), (\phi, \ell) \in [0, 2\pi) \times \mathbb{Z}$.

3.3 Associativity via the Floor Function

We sketch the proof that the operation $\circ$ is associative using only elementary properties of the floor function.

Theorem 1 *The operation $\circ$ is associative on $SO(2) \times \mathbb{Z}$.*

Proof Let (θ_j, k_j), $j = 1, 2, 3$, be arbitrary. Set

$$N_{12} := N(\theta_1, \theta_2), \quad N_{23} := N(\theta_2, \theta_3), \quad N_{123} := \left\lfloor \frac{\theta_1 + \theta_2 + \theta_3}{2\pi} \right\rfloor.$$

First,

$$(\theta_1, k_1) \circ (\theta_2, k_2) = (\theta_1 \oplus \theta_2,\ k_1 + k_2 + N_{12}),$$

so

$$\big((\theta_1, k_1) \circ (\theta_2, k_2)\big) \circ (\theta_3, k_3) = \big((\theta_1 \oplus \theta_2) \oplus \theta_3,\ k_1 + k_2 + N_{12} + k_3 + N(\theta_1 \oplus \theta_2, \theta_3)\big).$$

Using $\theta_1 \oplus \theta_2 = \theta_1 + \theta_2 - 2\pi N_{12}$, we have

$$N(\theta_1 \oplus \theta_2, \theta_3) = \left\lfloor \frac{\theta_1 + \theta_2 + \theta_3}{2\pi} - N_{12} \right\rfloor = N_{123} - N_{12},$$

so

$$(\theta_1 \oplus \theta_2) \oplus \theta_3 = \theta_1 + \theta_2 + \theta_3 - 2\pi N_{123},$$

and the integer part simplifies to $k_1 + k_2 + k_3 + N_{123}$.

Similarly,

$$(\theta_2, k_2) \circ (\theta_3, k_3) = (\theta_2 \oplus \theta_3,\ k_2 + k_3 + N_{23}),$$

so

$$(\theta_1,k_1)\circ\big((\theta_2,k_2)\circ(\theta_3,k_3)\big) = \big(\theta_1\oplus(\theta_2\oplus\theta_3),\ k_1+k_2+k_3+N_{23}+N(\theta_1,\theta_2\oplus\theta_3)\big).$$

Again,

$$N(\theta_1,\theta_2\oplus\theta_3) = \left\lfloor \frac{\theta_1+\theta_2+\theta_3}{2\pi} - N_{23} \right\rfloor = N_{123} - N_{23},$$

so

$$\theta_1\oplus(\theta_2\oplus\theta_3) = \theta_1+\theta_2+\theta_3 - 2\pi N_{123},$$

and the integer part is again $k_1+k_2+k_3+N_{123}$. Hence the two associativity bracketing patterns yield identical results. □

3.4 Isomorphism with $(\mathbb{R},+)$

We now show that the bijection Φ becomes a group isomorphism when $SO(2)\times\mathbb{Z}$ is equipped with the transported operation $\circ$.

Theorem 2 *The map*

$$\Phi : (SO(2)\times\mathbb{Z},\circ) \longrightarrow (\mathbb{R},+), \qquad \Phi(\theta,k) = \theta + 2\pi k,$$

is a group isomorphism.

Proof We have already shown that Φ is bijective. It remains to check that it is a homomorphism. For $(\theta,k),(\phi,\ell)\in[0,2\pi)\times\mathbb{Z}$,

$$(\theta,k)\circ(\phi,\ell) = (\theta+\phi-2\pi N(\theta,\phi),\ k+\ell+N(\theta,\phi)).$$

Applying Φ,

$$\begin{aligned}\Phi\big((\theta,k)\circ(\phi,\ell)\big) &= \theta+\phi-2\pi N(\theta,\phi)+2\pi\big(k+\ell+N(\theta,\phi)\big)\\ &= (\theta+2\pi k)+(\phi+2\pi\ell)\\ &= \Phi(\theta,k)+\Phi(\phi,\ell).\end{aligned}$$

Thus Φ is a bijective homomorphism. □

This realizes $(SO(2)\times\mathbb{Z},\circ)$ as a concrete algebraic model of the universal covering group of $SO(2)$.

4 The Universal Cover of $SE(2)$

The universal cover of the planar Euclidean motion group, $\widetilde{SE(2)}$ has group law

$$(x_1,t_1)(x_2,t_2) = (x_1+R(t_1)x_2, t_1+t_2), \tag{3}$$

where $R(t) \in SO(2)$ is the planar rotation matrix of angle t, and $x_i \in \mathbb{R}^2$ are translations. The (left and right) Haar measure is $dt\,dx$. The big difference is that $t_i \in \mathbb{R}$ without restriction.

The deck (covering) subgroup is

$$\Gamma = \{(0, 2\pi k) : k \in \mathbb{Z}\}. \tag{4}$$

Since $R(2\pi k) = I$, this subgroup lies in the center of $\widetilde{SE}(2)$. The quotient $\widetilde{SE}(2)/\Gamma$ is canonically isomorphic to $SE(2)$.

4.1 Exponential and Logarithm Maps for the Universal Cover of the Euclidean Motion Group of the Plane

The exponential map for $\widetilde{SE}(2)$ is essentially the same as for $SE(2)$, except that the rotational part of the Lie algebra element remains unchanged rather than going to a rotation matrix. And the logarithm is defined on almost the whole Lie algebra, minus a set of measure zero.

Exponential Map

Denote Lie algebra elements as $(v, \omega) \in se(2)$. Then

$$\exp(v,\omega) = \begin{cases} \left(\frac{1}{\omega} \begin{pmatrix} \sin\omega & -(1-\cos\omega) \\ 1-\cos\omega & \sin\omega \end{pmatrix} v,\ \omega \right), & \omega \neq 0, \\ (v, 0), & \omega = 0. \end{cases}$$

Logarithm Map

$$\log(a,\theta) = \begin{cases} \left(\frac{1}{2(1-\cos\theta)} \begin{pmatrix} \sin\theta & 1-\cos\theta \\ -(1-\cos\theta) & \sin\theta \end{pmatrix} a,\ \theta \right), & \theta \notin \{2\pi k \mid k \in \mathbb{Z}\}, \\ (a, 0), & \theta = 0. \end{cases}$$

Therefore we can write

$$\log(\exp(v,\omega)) \;=\; (v,\omega)$$

for all elements of the Lie algebra except for the set of measure zero corresponding to the edge cases $(v, 2\pi k)$ where $k \in \mathbb{Z} - \{0\}$. This is very different than the $SE(2)$ case in which ω is restricted to the open set $(-\pi, \pi)$ and hence (v, ω) is restricted to the corresponding strip in $se(2)$.

4.2 Uncertainty Modeling

In the fields of Localization, Mapping, and Navigation, modeling uncertainty is central. In this regard it is much more convenient to consider probability densities on $\widetilde{SE(2)}$ of the form $\tilde{f}(x, y, \theta)$ rather than their θ-folded counterparts on $SE(2)$ of the form

$$f(x, y, \theta) \;=\; \sum_{k \in \mathbb{Z}} \tilde{f}(x, y, \theta + 2\pi k)$$

In filtering theory, uncertainty is propagated and then in the measurement step is taken. Folding and group convolution commute and so there is no problem in propagating uncertainty in either model. But in Bayesian calculations, ratios of probability densities are computed, and this is not compatible with taking independent sums in the numerator and denominator. Moreover, there is no form closure after taking sums. In this regard, an unfolded density such as a Gaussian is a much more convenient object to deal with than a folded density. This is yet another reason to work in the universal cover.

For completeness, the commutativity of convolution and wrapping over θ are explained here.

4.3 Convolution on $\widetilde{SE(2)}$

For $\tilde{f}_1, \tilde{f}_2 \in L^1(\widetilde{SE(2)})$, convolution is defined by

$$(\tilde{f}_1 \tilde{*} \tilde{f}_2)(g) = \int_{\widetilde{SE(2)}} \tilde{f}_1(h)\, \tilde{f}_2(h^{-1} g)\, dh. \tag{5}$$

When $\tilde{f}_1$ and $\tilde{f}_2$ are probability density functions (pdfs), $\tilde{f}_1 * \tilde{f}_2$ is the pdf of the product of two independent $\widetilde{SE(2)}$-valued random variables.

4.4 Periodization

Define the periodization (pushforward) operator $P : L^1(\widetilde{SE(2)}) \to L^1(SE(2))$ by

$$(P\tilde{f})(x, t \mod 2\pi) = \sum_{k\in\mathbb{Z}} \tilde{f}(x, t + 2\pi k), \tag{6}$$

whenever the sum converges absolutely almost everywhere. This is precisely the pushforward of f under the covering homomorphism $\pi : \widetilde{SE(2)} \to SE(2)$.

4.5 Commutativity of Convolution and Periodization

Proposition 1 *Let $\tilde{f}_1, \tilde{f}_2 \in L^1(\widetilde{SE(2)})$ such that the above periodization sums are absolutely convergent almost everywhere and justify interchange of summation and integration. Then*

$$P(\tilde{f}_1 \tilde{*} \tilde{f}_2) = (P\tilde{f}_1) * (P\tilde{f}_2), \tag{7}$$

where the convolution on the right-hand side is taken on $SE(2)$.

Proof The covering map $\pi : \widetilde{SE(2)} \to SE(2)$ is a surjective group homomorphism with discrete, central kernel Γ. For such homomorphisms, pushforward intertwines convolution:

$$\pi_\#(\tilde{f}_1 \tilde{*} \tilde{f}_2) = (\pi_\# \tilde{f}_1) * (\pi_\# \tilde{f}_2) \tag{8}$$

Identifying $\pi_\#$ with the periodization operator P yields the claim.

Explicitly, expanding $P(\tilde{f}_1) * P(\tilde{f}_2)$ produces a double sum over Γ. Centrality of Γ implies $(h\gamma)^{-1}g = \gamma^{-1}h^{-1}g$, allowing reindexing of the summation. Fubini's theorem then collapses the double sum into the single sum defining $P(\tilde{f}_1 \tilde{*} \tilde{f}_2)$. □

5 Conclusions

The concepts of total orientation and total pose in the plane are explored here. It is shown that they are actually simpler structures than $SO(2)$ and $SE(2)$ and that they are fully backwards compatible with them, indicating that nothing is lost by using them. Moreover, it is shown that the the set $SO(2) \times \mathbb{Z}$ with an appropriate group operation (which is not the direct product) can be used in place of $(\mathbb{R}, +)$, which has the potential to extend to the three-dimensional case.

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